SIMULTANEOUS APPROXIMATION BY
SUMMATION – INTEGRAL TYPE OPERATORS

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Abstract - In this paper, we consider a general sequence of summation-integral type operators. The aim of the paper is to study some direct result of summation-integral type operators for functions of bounded variation.

I. INTRODUCTION

In the year 2003 Srivastava and Gupta[4] investigated as well as estimated the rate of convergence of the general sequence of operators \( G_{n,c} \) by means of the decomposition technique for functions of bounded variation. Also Ispir and Yükesel[2] introduced the Bezier variant of these operators and estimated the rate of convergence for function of bounded variations.

Srivastava and Gupta defined a summation-integral type operators \( G_{n,c} \) as follows,

\[
G_{n,c}(t,x) = \sum_{k=1}^{\infty} \int_0^1 P_{n,k-1}(t,c)f(t)dt + P_{n,0}(x,c)f(0)
\]

(1)

where \( P_{n,k}(x,c) = \frac{(-x)^k}{k!} \phi_{n,c}(x) \)

and \( \phi_{n,c}(x) \) is completely monotone so that

\[
(-1)^k \phi_{n,c}^{(k)}(x) \geq 0 \quad (0 \leq x \leq b)
\]

Here \( \{\phi_{n,c}(x)\}_{n=1}^{\infty} \) is a sequence of functions defined on the closed interval

\[ [0, b], \; b > 0 \]

which satisfy the following properties for every \( n \in \mathbb{N} \) and \( k \in N_0 = N \cup \{0\} \).

(i) \( \phi_{n,c} \in C^\infty[a,b] \) \quad (b > a \geq 0)

(ii) \( \phi_{n,c}(0) = 1 \)

(iii) \( \phi_{n,c}(x) \) is completely monotone so that

\[
(-1)^k \phi_{n,c}^{(k)}(x) \geq 0 \quad (0 \leq x \leq b)
\]

(iv) There exist an integer \( c \) such that,

\[
\phi_{n,c+1}(x) = -n \phi_{n,c}^{(k)}(x)
\]

(\( n > \max\{0,-c\}; \; x \in [0,b] \))

2) CONSTRUCTION OF THE OPERATORS

In this section, the operators \( G_{n,c}(f,x) \) defined by (1) can also be considered when \( c = -1 \), we have

\[
G_{n,-1}(f,x) = \sum_{k=1}^{n} \int_0^1 P_{n,k}(x-1)f(t)dt + (1-x)^n f(0)
\]

(2)

where, \( P_{n,k}(x;1) = \binom{n}{k} x^k (1-x)^{n-k} \)

On the other hand, the general operators defined by (2) can alternatively be written in the form,

\[
G_{n,-1}(f;x) = \int_0^1 f(t)dt
\]

(3)

where, \( k_{n}(x,t;1) = \binom{n}{k} x^k (1-x)^{n-k} \)

(4)

In the present paper, we estimated direct results of the operator \( G_{n,-1} \) by means of the decomposition technique for functions of bounded variation using auxiliary function \( g_x(t) \) which is defined by

\[
g_x(t) = \begin{cases} f(t) - f(x) & (0 \leq t < x) \\ 0 & (t = x) \\ f(t) - f(x) & (x < t < \infty) \end{cases}
\]

(5)

3) AUXILLIARY RESULTS

In order to prove our main result we require following lemmas.

Lemma -1[3]. For all \( x \in (0,\infty) \) and \( k \in \mathbb{N} \),

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\[ P_{n,k} (x,-1) \leq \frac{1}{\sqrt{2e}} \frac{1}{\sqrt{nx(1-x)}} \]  

\textbf{Lemma-2\textsuperscript{4}}: Let \[ \mu_{n,k}(x) = \sum_{k=0}^{n} \frac{1}{2} \int_{0}^{\mu_{n,k-1}(t)} f(x-y) \, dy \] \[ \mu_{n,k}(x) = \frac{1}{(n+1)(n+2)} x^n \]  

Then, \( \mu_{n,0}(x,-1) = 1 \), \( \mu_{n,1}(x,-1) = \frac{-x}{n+1} \) and \( \mu_{n,2}(x,-1) = \frac{x(1-x)}{(n+1)(n+2)} \)  

In particular, given any number \( x > 0 \) lemma 2 yields the inequality, \[ \mu_{n,2}(x,-1) \leq \frac{\lambda x(1-x)}{n \lambda} \]  

\textbf{Lemma -3\textsuperscript{4}}. Let \( x \in (0,1) \) and \( k_n(x,t;\,1) \) be defined by (5). Then for \( \lambda > 2 \) and for sufficiently large \( n \),  

\[ B_n(x,y) = \int_{0}^{y} k_n(x,t;\,1) \, dt \leq \frac{\lambda x(1-x)}{n(x-y)^2} \]  

\[ (0 \leq y < x) \]  

and  

\[ 1 - B_n(x,z) = \int_{z}^{x} k_n(x,t;\,1) \, dt \leq \frac{\lambda x(1-x)}{n(z-x)^2} \]  

\[ (x < z \leq 1) \]  

\textbf{Proof}: Since \( 0 \leq y < x \), for \( t \in [0, y] \) we have,  

\[ \frac{x-t}{x-y} \geq 1 \]  

from definition (4) we find that,  

\[ \int_{0}^{y} k_n(x,t;\,1) \, dt \leq \frac{\lambda x(1-x)^2}{n(x-y)^2} \]  

\[ (0 \leq y < x) \]  

The proof of inequality (11) is similar.  

\textbf{4) MAIN RESULTS}  

In this section, we prove the following result.  

\textbf{Theorem}: Let \( f \) be a function of bounded variation on every finite sub-interval of the closed interval \([0,1]\). Suppose also that the one-sided limits \( f(-) \) and \( f(+) \) exist for some fixed point \( x \in (0,1) \). Then, for \( \lambda > 2 \) and sufficiently large \( n \),  

\[ \left| G_{x,-1}(t-x,\,1) \right| = \frac{1}{2} \sqrt{\left| f(x)-f(x) \right|} \leq \frac{1}{\sqrt{4e\sqrt{n-1}} \sqrt{x-1} \sqrt{x-1}} \]  

\[ \sum_{k=1}^{n} V_{x-1,x}^{\sqrt{k}} (g(x)) \]  

where, \( g_x(t) = \begin{cases} f(t) - f(x) & (0 \leq t < x) \\ f(t) - f(x+1) & (x < t < 1) \end{cases} \)  

and \( V_{a,b}^{g_x} \) denotes total variation of \( g_x \) on \([a, b]\).  

\textbf{Proof}: In our proof of theorem first.  

\[ \left| G_{x,-1}(t-x,\,1) \right| \leq \frac{1}{2} \sqrt{\left| f(x)-f(x) \right|} \leq \frac{1}{2} \sqrt{\left| f(\xi_0)-f(x) \right|} \leq \frac{1}{2} \sqrt{\left| f(x)-f(x) \right|} \]  

(14) \]  

We need estimates for \( G_{n,1}(\text{sign}(t-x),\,x) \) and \( G_{n,1}(\text{sign}(t-x),\,x) \).  

To estimates \( G_{n,1}(\text{sign}(t-x),\,x) \), we first observe that,  

\[ G_{n,1}(\text{sign}(t-x),\,x) = \int_{0}^{1} k_n(x,t;\,1) \, dt \]  

\[ = \int_{0}^{1} k_n(x,t;\,1) \, dt - \int_{0}^{1} k_n(x,t;\,1) \, dt \]  

\[ = 2 \int_{0}^{1} k_n(x,t;\,1) \, dt - 1 \]  

\[ \sin \left( \int_{0}^{1} k_n(x,t;\,1) \, dt \right) = 1 \]  

\[ (0 \leq y < x) \]  

The proof of inequality (11) is similar.  

The proof of inequality (11) is similar.
Next to estimate of $G_{n-1}(g_x,x)$ we have,

$$G_{n-1}(g_x,x) = \int_0^x g_x(t) k_n(x,t;1) dt$$

$$= \left[ \int_0^{x-x/n} + \int_{x-x/n}^{x-1} + \int_{x-1}^{1} \right] k_n(x,t;1) g_x(t) dt = E_1 + E_2 + E_3 \quad \text{... (18)}$$

For $t \in \left[ x - x/n - x + (1-x)/\sqrt{n} \right]$ we have,

$$|g_x(t)| \leq V_{x}^{x+1}(1-x)/\sqrt{n} (g_x) \leq \frac{1}{n} \sum_{k=1}^{n} V_{x}^{x+1}(1-x)(g_x)$$

and so

$$|E_2| \leq V_{x}^{x+1}(1-x)/\sqrt{n} (g_x) \leq \frac{1}{n} \sum_{k=1}^{n} V_{x}^{x+1}(1-x)(g_x) \quad \text{... (19)}$$

In order to estimate $E_1$, we set $y = x - x/\sqrt{n}$ and integrate by parts; we thus obtain

$$E_1 = \int_0^y g_x(t) dt (B_n(x,t)) = g_x(y) B_n(x,y) - \int_0^y B_n(x,t) dt (g_x(t))$$

Since $|g_x(y)| \leq V_y (g_x)$ conclude that

$$|E_1| \leq V_y (g_x) B_n(x,y) + \int_0^y B_n(x,t) dt (V_t (g_x))$$

For $y = x - x/\sqrt{n}$ by using lemma (3), we get,

$$|E_2| \leq \frac{\lambda x (1-x)}{n(x-y)^2} V_y (g_x) + \frac{\lambda x (1-x)}{n} \int_0^y (x-t)^2 dt (V_t (g_x))$$

Integrating the last integral by parts, we get

$$|E_1| \leq \frac{\lambda x (1-x)}{n} \left[ x^2 V_y (g_x) + 2 \frac{V_y (g_x)}{x} \right] \quad \text{... (20)}$$

for which $y = x - x/\sqrt{n}$, yields

$$\int_0^{x-x/n} V_x (g_x)(x-t)^3 dt = \sum_{k=1}^{n} \int_{x-x/n}^{x-x/n/k} V_x (g_x)(x-t)^3 dt \leq \frac{1}{2c} \sum_{k=1}^{n} V_{x}^{x+1}(1-x/k) (g_x)$$

Hence, $|E_1| \leq \frac{\lambda (1-x)}{nx} \sum_{k=1}^{n} V_{x}^{x+1}(1-x/k) (g_x) \quad \text{... (21)}$

Using a similar method and lemma, we obtain,

$$|E_3| \leq \frac{\lambda (1-x)}{nx} \sum_{k=1}^{n} V_{x}^{x+1}(1-x/k) (g_x) \quad \text{... (22)}$$

From equations (19), (21) and (22), it follows that

$$|G_{n-1}(g_x,x)| \leq \frac{\lambda (1-x)}{nx} \sum_{k=1}^{n} V_{x}^{x+1}(1-x/k) (g_x) + \frac{\lambda (1-x)}{nx} \sum_{k=1}^{n} V_{x}^{x+1}(1-x/k) (g_x)$$

$$\leq \frac{\lambda (1-x)}{nx} + \frac{\lambda (1-x)}{nx} \sum_{k=1}^{n} V_{x}^{x+1}(1-x/k) (g_x) \leq \frac{2\lambda (1-x)}{nx} \sum_{k=1}^{n} V_{x}^{x+1}(1-x/k) (g_x)$$

Our theorem now follows from (19), (21) and (23). This completes the proof for the theorem.

REFERENCES:

