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LOCAL FRACTIONAL VARIATIONAL ITERATION METHOD FOR SOLVING NONLINEAR DIFFERENTIAL EQUATIONS

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***Abstract:** In this article, the local fractional variational iteration method is employed to obtain approximate analytical solution to differential fractional equations of Bernoulli. Some examples are given to illustrate the efficiency and accuracy of the proposed method to obtain analytical solutions to differential equations within the local fractional derivatives.*

***Keywords* –** LFVIM method, local fractional derivative, local fractional integral, differential fractional equation.

I. INTRODUCTION

Recently the local fractional variational iteration method [2] has been widely applied to analytically solve fractional differential equations. The method is derived from local fractional operators ([9], [10], [11], [12], [13], [14], [15], [16]).

which accurately computes the solutions in a local fractional or in an exact form, presents interest to applied sciences for problems where the other methods cannot be applied properly.

The structure of the paper is as follows. In Section 2, we give the concept of local fractional calculus. In Section 3, we give analysis of the local fractional variational iteration method. In Section 4, the proposed method is implemented to obtain approximate analytical solution to differential fractional equations of Bernoulli. In Section 5, we consider some illustrative examples. Finally, we present our conclusions.

II. PRELIMINARIES

In this section, we introduce some definitions and properties that will be used later.

Definition 1 *The local fractional derivative of* $f(x)$ *of order* α *at* $x=x_0$ *is given by*

$$\frac{d^\alpha}{dx^\alpha}f(x)\Big|_{x=x_0} = f^{(\alpha)}(x) = D^\alpha f(x) = \lim_{x\to x_0}\left(\frac{\Delta^\alpha\big(f(x)-f(x_0)\big)}{(x-x_0)^\alpha}\right) \tag{2.1}$$

where

$$\Delta^\alpha\big(f(x)-f(x_0)\big) = \Gamma(\alpha+1)\big(f(x)-f(x_0)\big)$$

Definition 2 *The local fractional integral of* $f(x)$ *of order* α *in the interval [a,b] is given by*

$$ {}_aJ_b^\alpha f(t) = \frac{1}{\Gamma(\alpha+1)}\int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(\alpha+1)}\lim_{\Delta t\to 0}\left(\sum_{j=0}^{m-1} f(t_j)(\Delta t_j)^\alpha\right) \tag{2.2}$$

where the partitions of the interval $[a,b]$ *are denoted as* (t_j, t_{j+1}) *with* $\Delta t_j = t_{j+1}-t_j$, $t_0 = a$, $t_m = b$ *and* $\Delta t = max(\Delta t_0, \Delta t_1, \dots, \Delta t_{m-1})$, $j = 0,1,\dots,m-1$.

Definition 3 *The Mittage Leffler function is defined as*

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{x^{k\alpha}}{\Gamma(k\alpha+1)}, \qquad 0<\alpha\le 1 \tag{2.3}$$

According to local fractional derivative and local fractional integral , we have:

$$D^\alpha\big(\lambda f(x)\big) = \lambda D^\alpha\big(f(x)\big) \tag{2.4}$$

$$ {}_aJ_b^\alpha\big(\lambda f(x)\big) = \lambda\ {}_aJ_b^\alpha\big(f(x)\big) \tag{2.5}$$

$$\frac{d^\alpha}{dx^\alpha}\left(\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}\right) = \frac{x^{(n-1)\alpha}}{\Gamma\big((n-1)\alpha+1\big)}, \quad n\in\mathbb{N} \tag{2.6}$$

$$ {}_0J_x^\alpha\left(\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\right) = \frac{x^{(n+1)\alpha}}{\Gamma\big((n+1)\alpha+1\big)}, \quad n\in\mathbb{N} \tag{2.7}$$

For more details, we refer the interested reader to ([2], [3], [4], [5], [6], [7], [8]).

III. Analysis of the Local Fractional Variational Iteration Method

The local fractional variational iteration method structured in [2] was applied to deal with the local fractional diferential equations arising in mathematical physics [3, 4, 5, 6, 7, 8].

To clarify the basic ideas of LFVIM, we consider the following nonlinear local fractional differential equations:

$$L_\alpha u(t) + N_\alpha u(t) = f(t) \tag{3.1}$$

where L_α is the linear operator, N_α is the nonlinear operator and *f(t)* is inhomogeneous term.

According to LFVIM, we can write down a correction local fractional functional as follows:

$$u_{n+1}(t) = u_n(t) + \frac{1}{\Gamma(\alpha+1)}\int_0^t \left\{\frac{\lambda(\xi)^\alpha}{\Gamma(\alpha+1)}\left(L_\alpha(u(\xi)) + N_\alpha(\tilde{u}(\xi)) - f(\xi)\right)\right\}(d\xi)^\alpha \tag{3.2}$$

where $\frac{\lambda(\xi)^\alpha}{\Gamma(\alpha+1)}$ is a fractal Lagrange multiplier.

Taing the local fractional variation of Eq(3.2) with respect to the independent variable we find that

$$\delta^\alpha u_{n+1}(t) = \delta^\alpha u_n(t) + \delta^\alpha \frac{1}{\Gamma(\alpha+1)}\int_0^t \left\{\frac{\lambda(\xi)^\alpha}{\Gamma(\alpha+1)}\left(L_\alpha(u(\xi)) + N_\alpha(\tilde{u}(\xi)) - f(\xi)\right)\right\}(d\xi)^\alpha \tag{3.3}$$

The extremum condition of u_{n+1} requires that $\delta^\alpha u_{n+1} = 0$. This yields the stationary conditions

$$1 - \left(\frac{\lambda(\xi)^\alpha}{\Gamma(\alpha+1)}\right)^{(\alpha)}\Bigg|_{\xi=t} = 0$$
$$\frac{\lambda(\xi)^\alpha}{\Gamma(\alpha+1)}\Bigg|_{\xi=t} = 0$$
$$\left(\frac{\lambda(\xi)^\alpha}{\Gamma(\alpha+1)}\right)^{(2\alpha)}\Bigg|_{\xi=t} = 0 \tag{3.4}$$

So, from (3.5), we get

$$\frac{\lambda(\xi)^\alpha}{\Gamma(\alpha+1)} = \frac{(\xi-t)^\alpha}{\Gamma(\alpha+1)} \tag{3.5}$$

The function $u_0(t)$ should be selected by using the initial conditions as follows

$$u_0(t) = u(0) + \frac{t^\alpha}{\Gamma(\alpha+1)}u^{(\alpha)}(0) \tag{3.6}$$

We can obtain a correction local fractional functional, which reads

$$u_{n+1}(t) = u_n(t) + \frac{1}{\Gamma(\alpha+1)}\int_0^t \left\{\frac{(\xi-t)^\alpha}{\Gamma(\alpha+1)}\left(L_\alpha(u(\xi)) + N_\alpha(\tilde{u}(\xi)) - f(\xi)\right)\right\}(d\xi)^\alpha \tag{3.7}$$

Consequently, the solution is obtained as:

$$u(t) = \lim_{n\to+\infty}(u_n(t)) \tag{3.8}$$

IV. Application of LTVIM Method

Consider the problem

$$D_t^\alpha y(t) + a(t)y(t) = b(t)y^m(t), \quad 0 < \alpha \le 1, m \in \mathbb{N}, \tag{4.1}$$

where the operator D stand for the local fractional derivative and *f,g* are continuous functions on the real line.

Take the initial condition as

$$y(0) = \lambda \tag{4.2}$$

By using Eq. (3.7) we structure a local fractional iteration procedure as

$$u_{n+1}(t) = u_n(t) + \frac{1}{\Gamma(\alpha+1)}\int_0^t \left\{\frac{(\xi-t)^\alpha}{\Gamma(\alpha+1)}\begin{pmatrix} D_\xi^\alpha(y_n(\xi)) \\ +a(\xi)\widetilde{y_n}(\xi) - b(\xi)\left(\widetilde{y_n}(\xi)\right)^m\end{pmatrix}\right\}(d\xi)^\alpha \tag{4.3}$$

The initial value $u_0(t)$ is given by

$$y_0(t) = y(0) + \frac{t^\alpha}{\Gamma(\alpha+1)}y^{(\alpha)}(0) \tag{4.4}$$

Hence, we can derive the first approximation term

$$u_1(t) = u_0(t) + \frac{1}{\Gamma(\alpha+1)}\int_0^t \left\{\frac{(\xi-t)^\alpha}{\Gamma(\alpha+1)}\begin{pmatrix} D_\xi^\alpha(y_0(\xi)) \\ +a(\xi)\widetilde{y_0}(\xi) - b(\xi)\left(\widetilde{y_0}(\xi)\right)^m\end{pmatrix}\right\}(d\xi)^\alpha \tag{4.5}$$

The second approximation term

$$u_2(t) = u_1(t) + \frac{1}{\Gamma(\alpha+1)}\int_0^t \left\{\frac{(\xi-t)^\alpha}{\Gamma(\alpha+1)}\begin{pmatrix} D_\xi^\alpha(y_1(\xi)) \\ +a(\xi)\widetilde{y_1}(\xi) - b(\xi)\left(\widetilde{y_1}(\xi)\right)^m\end{pmatrix}\right\}(d\xi)^\alpha \tag{4.6}$$

The third approximation term

$$u_3(t) = u_2(t) + \frac{1}{\Gamma(\alpha+1)}\int_0^t \left\{\frac{(\xi-t)^\alpha}{\Gamma(\alpha+1)}\begin{pmatrix} D_\xi^\alpha(y_2(\xi)) \\ +a(\xi)\widetilde{y_2}(\xi) - b(\xi)\left(\widetilde{y_2}(\xi)\right)^m\end{pmatrix}\right\}(d\xi)^\alpha \tag{4.7}$$

V. Illustrative Examples

Example 5.1

Consider the problem

$$D_t^\alpha y(t) - y(t) = 0, \quad 0 < \alpha \le 1 \tag{5.1}$$

Take the initial condition as

$$y(0) = 1, y^\alpha(0) = 1 \tag{5.2}$$

The exact solution of Eq(5.1) for the special case α=1 is

$$y(t) = e^t \tag{5.3}$$

In order to obtain numerical solution of equation (5.1), using the expression (4.3), we can obtain:

$$y_{n+1}(t) = y_n(t) + J_t^\alpha\left\{\frac{(\tau-t)^\alpha}{\Gamma(\alpha+1)}\left(D_\tau^\alpha(y_n(\tau)) - y_n(\tau)\right)\right\} \tag{5.4}$$

By the above iteration formula (5.4), we can obtain directly the other components as

$$y_0(t) = 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} \quad (5.5)$$

$$y_1(t) = 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \quad (5.6)$$

$$y_2(t) = 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{t^{5\alpha}}{\Gamma(5\alpha+1)} \quad (5.7)$$

The LTVIM gives the solution for Eq(5.1) in the case of α=1:

$$y(t) = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \cdots = \sum_{n\geq 0} \frac{t^n}{n!} = e^t \quad (5.8)$$

Example 5.2

Consider the problem

$$D_t^\alpha y(t) + 2y(t) = 4y^3(t), \quad 0 < \alpha \leq 1, \quad (5.9)$$

Take the initial condition as

$$y(0) = \frac{1}{2}, y^\alpha(0) = -\frac{1}{2}, \quad (5.10)$$

The exact solution of Eq(5.9) for the special case α=1 is

$$y(t) = \frac{1}{\sqrt{2+2e^{4t}}} \quad (5.11)$$

In order to obtain numerical solution of equation (5.9), using the expression (4.3), we can obtain:

$$y_{n+1}(t) = y_n(t) + J_t^\alpha\left\{\frac{(\tau-t)^\alpha}{\Gamma(\alpha+1)}\left(D_\tau^\alpha\left(y_n(\tau)\right) + 2y_n(\tau) - 4{y_n}^3(t)\right)\right\} \quad (5.12)$$

By the above iteration formula (5.12), we can obtain directly the other components as

$$y_0(t) = \frac{1}{2} - \frac{1}{2}\frac{t^\alpha}{\Gamma(\alpha+1)} \quad (5.13)$$

$$y_1(t)\ \frac{1}{2} - \frac{1}{2}\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{1}{2}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{1}{2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{3}{2}\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{1}{2}\frac{t^{5\alpha}}{\Gamma(5\alpha+1)} \quad (5.14)$$

The LTVIM gives the solution for Eq (5.9) in the case of α=1:

$$y(t) = \frac{1}{2} - \frac{1}{2}t - \frac{1}{2}t^2 + \cdots \quad (5.15)$$

VI. CONCLUSIONS

The local fractional variational iteration method has been applied to differential fractional equations of Bernoulli in order to find its approximate analytical solutions. The results show that the applied method is suitable and inexpensive for obtaining the approximate solutions.

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