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ANALYSIS OF THE CHAOTIC NATURE IN A CUBIC MAP

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Abstract: In this paper we consider a cubic map $ax^3 + bx^2 + cx + d = 0$, where a,b,c and d are parameters. A cubic map is a bimodal map which is an iterated discrete-time dynamical system exhibiting chaotic behavior having infinitely many unstable periodic points. Here we have proved the following:
This map exhibits transition to period doubling route and finally leads to a bounded chaotic region that possesses many exciting dynamical properties to be studied .
Its periodic doubling bifurcation and the reverse bifurcation are studied for different values of the parameters.
An analysis has been carried out by using Lyapunov Exponent and Time Series technique .
Feigenbaum Universal Delta along with accumulation point have been calculated in order to establish Feigenbaum universal Properties.
Many open problems have been posed .

Keywords **–** Period doubling Bifurcation, Reverse bifurcation, Lyapunov Exponent, Feigenbaum Delta, Chaos, Time Series.

AMS Subject Classification:37G35, 37B10, 37D45

1: INTRODUCTION

Dynamical System is the study of the long-term periodic behavior of evolving systems. The modern theory of dynamical systems was originated at the end of 19^{th} century with many fundamental questions concerning the stability and evolution of the solar system. Attempts to answer these questions have led to the development of a rich and powerful field with applications to physics, biology, meteorology, astronomy, economics and others, [1,2,5,10].

At the same time dynamical systems often appear to be chaotic in the sense that they have sensitive dependence to initial conditions , that is, minor changes in the initial state lead to a dramatically different long term behavior leading to a Chaotic region. In Mathematics, researchers deal with nonlinear maps to study the various qualitative features related to its chaotic nature.

2. The Main Study and Results:

2.01 **Period Doubling and Undoubling Bifurcation :**

A bifurcation is a sudden change in the qualitative behavior of a dynamical system which takes place with the variation of the control parameter. The bifurcation then refers to the splitting of the behavior of the system into two regions: one above, the other below the particular parameter value at which the change occurs.

A period doubling bifurcation in a discrete dynamical system is a bifurcation in which the system switches to a new behaviour with twice the period of the original system. Many dynamical systems depend on parameters. Normally a gradual variation of a parameter in the system corresponds to a gradual variation of the solutions of the problem. However, there exists a large number of problems for which the number of solutions changes abruptly and structure of solution manifolds varies dramatically when a parameter passes through some critical values. This kind of phenomenon is called bifurcation and these parameter values are called bifurcation values or bifurcation points. As it is seen with other dynamical systems, the nature of the fixed points of iterated maps can change as the control parameters of the system change. In this part we will focus on the sequence of period- doublings that leads to chaotic behavior, [1,2,6,8].

A cubic model is described by $ax^3 + bx^2 + cx + d = 0$

where a,b,c,d are parameters. For experimental convenience we fix the values of the parameters as
a= 0.35, b=-2.75 and d=0. The parameter c is allowed to vary.

The cubic map is then written as

$$f[x] = 0.35x^3 - 2.75x^2 + cx$$

The critical points are obtained by equating

$0.35x^3 - 2.75x^2 + cx = x$

$\Rightarrow 0.35x^3 - 2.75x^2 + (c-1)x = 0$

$\Rightarrow x(0.35x^2 - 2.75x + (c-1)) = 0$

The critical points are x=0 and

$$0.35x^2 - 2.75x + (c-1) = 0$$

$$x = \frac{2.75 \pm \sqrt{2.75^2 - 4 \times 0.35(c-1)}}{2 \times 0.35}$$

The other two fixed points depend on the values of c.

We examine the stability of the fixed points x = 0 and

$$x = \frac{2.75 \pm \sqrt{8.9625 - 1.4c}}{0.7}$$

$$\left|\frac{df}{dx}\right|_{x=0} = c$$

$$\left|\frac{df}{dx}\right|_{x=\frac{2.75 \pm \sqrt{7.5625 - 1.4(c-1)}}{0.7}} = 13.8036 \pm 3.92857\sqrt{8.9625 - 1.4c} - 2c$$

At c=3.326

$$\left|\frac{df}{dx}\right|_{x=\frac{2.75 \pm \sqrt{7.5625 - 1.4(c-1)}}{0.7}} = -1$$

From the equation the fixed point at x=0 is a stable (attracting) if 0< c <1

and is an unstable (repelling) fixed point if c >1 . The fixed point at $x = \frac{2.75 \pm \sqrt{7.5625 - 1.4(c-1)}}{0.7}$ is stable or

unstable according as 1<c <3.326. Thus if c >1, the fixed point at x=0 is an unstable fixed point . For 1<c<3.326, the fixed point at $x = \frac{2.75 + \sqrt{7.5625 - 1.4(c-1)}}{0.7}$ becomes unstable whereas at $x = \frac{2.75 - \sqrt{7.5625 - 1.4(c-1)}}{0.7}$ becomes stable.

The trajectory behavior is more interesting for c >3.326. For c just greater than 3.326, the trajectories settle into a pattern of alteration between two points. These two fixed points are attracting fixed points of a "two cycle". Thus we say that at c=3.326, the cubic map trajectories undergo a period doubling bifurcation. Below c=3.326, the trajectories converges to a single value of x, [4,5,9].

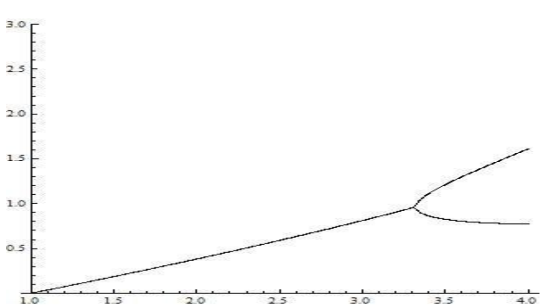


Fig. 1: period doubling at c=3.326

To find the second bifurcation we consider the second iterate $f^2(x)$ of the cubic function. The fixed point $f^2(x)$ is given by

$$\boldsymbol{f^2(x) = x}$$

The two cycle points x_1^* and x_2^* are the fixed points of the second iterate function:

$$x_1^* = f^{(2)}(x_1^*) \quad , \quad x_2^* = f^{(2)}(x_2^*)$$

For c just greater than 3.328, these fixed points of the second iterate becomes stable fixed points. The derivative of the map function and of the second iterate function that changes at the bifurcation value, tells us that the derivative of the map function passes through -1 as the value of c increases. We can evaluate the derivative of the second iterate function by using Chain rule of differentiation.

$$\frac{d f^{(2)}(x)}{dx} = \frac{d f(f(x))}{dx}$$

$$\frac{df}{dx}\Big|_{f(x)} \frac{df}{dx}\Big|_{x}$$

If we now calculate the derivative at one of the fixed points say x_1^*, we find

$$\frac{d f^{(2)}(x)}{dx}\Big|_{x_1^*} = \frac{df}{dx}\Big|_{x_2^*} \frac{df}{dx}\Big|_{x_1^*}$$

$$= \frac{d f^{(2)}(x)}{dx}\Big|_{x_2^*}$$

Thus the derivative of $f^2(x)$ are the same at both the fixed points , which are part of both upper and lower part of two cycles. That is, both of these fixed points either are stable or both are unstable and that they have the same "degree" of stability or instability. As c increases further the derivative of $f^{(2)}$ decreases and the fixed point become stable, where the slope of $f^{(2)}$ is less than 1 at these fixed points. The 2-cycle fixed points continue to be stable until c=4.086…..where the derivative of $f^{(2)}$ evaluated at the two cycle fixed points is equal to −1.Hence, for the parameter value just greater than c=4.086, the 2-cycle fixed points are unstable fixed points. Now, for the parameter value just greater than c=4.086, the trajectories settle into a 4-cycle, that is the trajectory cycles among 4 values. These x values are fixed points of the fourth iterate function $f^{(4)}$.[4,8,10,11]

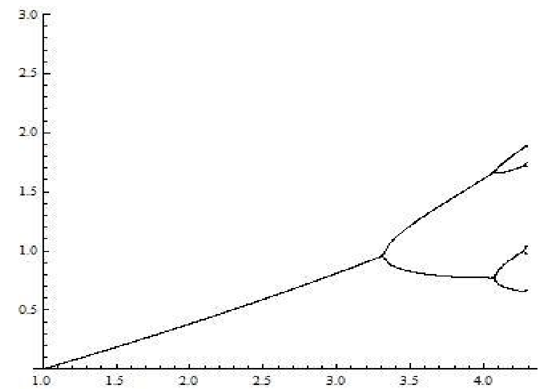


Fig. 2 Period Doubling at c=4.086

There is another period doubling bifurcation: The system has changed from 2-cycle behavior to 4-cycle behavior. Thus the 4-cycle is born when the derivative of $f^{(2)}$ evaluated at its 2-cycle fixed point passes through a value -1 and becomes more negative while the derivative of $f^{(4)}$. The process of period doubling repeats producing cycles of periodicity 2^3 ,2^4 ….. until we reach the limit point at c=c∞ , which is called the accumulation point of period doublings. At this point, the periodicity is 2^{∞} or iterates of the map become aperiodic.

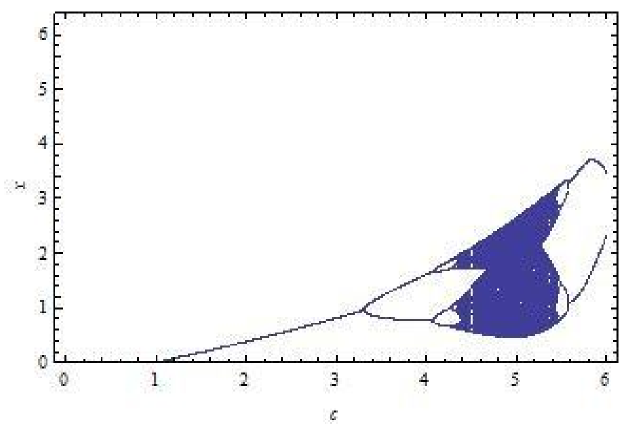


Fig. 3: Bifurcation Diagram

This period doubling route to chaos is called the Feigenbaum scenario.

Table-1:

Bifurcations Points	Periods of the given map
C_1=3.326..	2
C_2=4.086…	4
C_3=4.286…..	8
C_4=4.3257…..	16
C_5=4.3367…..	32
C_6=4.33875…..	64

Based on the bifurcation points a table of experimental Feigenbaum delta value is calculated by the following formula:

$$\delta_n = \frac{\mu_{n+1} - \mu_n}{\mu_{n+2} - \mu_{n+1}}$$

Table-2: Calculation of Feigenbaum Delta

Bifurcation Points	Feigenbaum delta
C_1=3.326	
C_2=4.086…..	
C_3=4.286…..	δ_1 =3.8
C_4=4.3257	δ_2 =5.03778
C_5=4.3367	δ_3 =3.60909
C_6=4.33875	δ_4 =5.36585
C_7=4.3391890475115	δ_5 =4.6691985399
C_8=4.3928307810543	δ_6 =4.6691985400
C_9=4.3393032165923	δ_7 =4.66919853953
C_{10}=4.339075296424	δ_8=4.669185405152
C_{11}=4.339308453366	δ_9=4.669185254980
C_{12}=4.339308651199	δ_{10}=4.66919864715
C_{13}=4.339308693569	δ_{11}=4.66918931282

It is observed that the Feigenbaum Delta converges to 4.669……

Remark: In the Fig. 3(c) it can be seen that for c > c_∞ the bifurcation diagram reveals an unexpected mixture of order and chaos, with periodic windows interspersed between chaotic clouds of dots. As the control parameter is moved further away from the periodic window, the bursts become more frequent until the system is fully chaotic. This progression is called the intermittency route to chaos. This process continues till c <5.463. At c=5.463……there is period 8 behavior, it then undergoes period 4, period 2 and ultimately converges to one point when c=6.365. That is there is period undoubling after passing through a chaotic region.

1. Determination of Accumulation Points :

Let $\{c_n\}$ be the sequence of the bifurcation points. Using Feigenbaum ð, if we know the first (c_1) and second(c_2) bifurcation points then we can expect third bifurcation point (c_3) as

$$c_3 \approx \frac{c_2 - c_1}{\delta} + c_2$$

The occurrence of first two period-doubling does not give assurance that a third will occur, but if it occurs, then it can be predicted by the above equation. Similarly

$$c_4 \approx \frac{c_3 - c_2}{\delta} + c_3$$

Which implies $c_4 \approx (c_2 - c_1)(\frac{1}{\delta} + \frac{1}{\delta^2}) + c_2$

Continuing this procedure to obtain c_5, c_6 and so on. We can sum the series to obtain the following result

$$c_\infty \approx \frac{(c_{n+1} - c_n)}{\delta - 1} + c_{n+1}$$

However this expression is exact when the bifurcation ratio $\delta_n = \frac{c_{n+1} - c_n}{c_{n+2} - c_{n+1}}$ is equal for all value

of n .

In fact { δ_n} converges as n $\rightarrow \infty$, that is . $\delta_n = \delta$

So, we consider the sequence $c_{\infty,n}$, $c_{\infty,n} \approx \frac{(c_{n+1} - c_n)}{\delta - 1} + c_{n+1}$

Where $c_{\infty,n}$ are the experimental value of bifurcation points . Clearly $\lim_{n \to \infty} c_{\infty,n} = c_\infty$

Using the experimental bifurcation points the sequence of accumulation points {$c_{\infty,n}$} is calculated for some values of *n*, as given below.

Table-3

$c_{\infty,1}$=4.3367	$c_{\infty,2}$=4.33875
$c_{\infty,3}$=4.3391890475115	$c_{\infty,4}$=4.3928307810543
$c_{\infty,5}$=4.3393032165923	$c_{\infty,6}$=4.339075296424
$c_{\infty,7}$=4.339308453366	$c_{\infty,8}$=4.339308651199

The above sequence converges to the value 4.339308…. which is the required accumulation point.[4.6]

1. Time Series Analysis:

A type of plot that is frequently used for visualization of the long term behavior of one dimensional difference equations of the form $x_{n+1} = f(x_n), n = 0,1,2$ …………… is called a time series plot. It consists of a representation of the variable x_n as a function of *n*.

The horizontal axis represents *n* (number of iteration) and the vertical axis represents x_n.

Below, we have shown the time series plot x_n vs. n of

$$x_{n+1} = 0.35x_n^3 - 2.75x_n^2 + c\, x_n$$

The initial condition is x=0.2 and the parameter c ≤ 3.28 .To make the sequence clear all the discrete points (*n*, x_n) are connected by line segments. Time series graph is of non sensitive, stable behavior and converges to a particular point showing period one behavior.

It is observed that the cubic map shows period one behavior when 1< c < 3.28. As c ≥ 3.286 the fixed point becomes unstable. It is found that when c ≤ 3.28 the curve converges to a particular point showing period one behavior. For c just greater than 3.28 i.e when c = 3.326 the curve oscillates between two fixed points showing period two behavior. Similarly for c=4.086 it shows period four behavior. At c = 4.3257 it shows period eight behavior. The cubic model shows chaotic behavior within the range of 4.4 < μ < 5.4 . Also for c ≥ 5.463 it shows period eight behavior. At c=5.541 it oscillates between four fixed points showing period four behavior. At c=5.775, period two appears finally when c=6.365 it shows period one behavior, [3,6,7,9].

Thus the cubic model shows period doubling when c<4.312 then it undergoes route to chaos till c<5.543.

The results are plotted here as a time series of x_n vs. n.

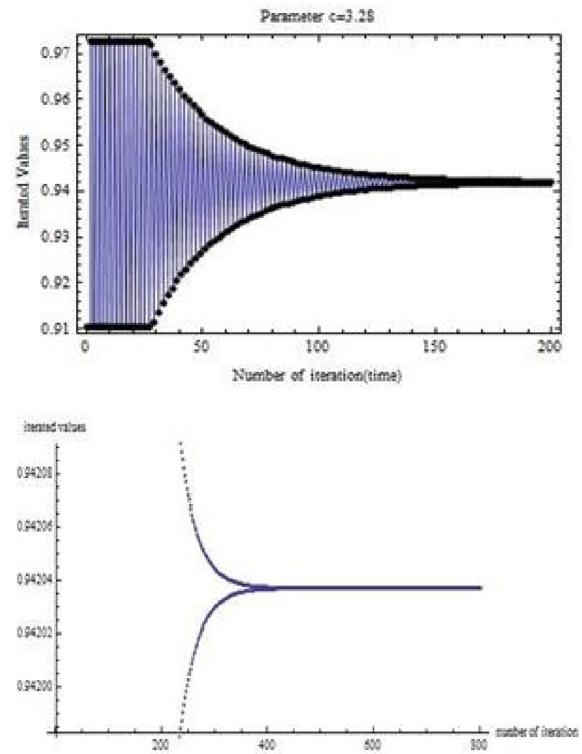


Fig. 4: Time series plot of xn vs. n at c=3.28

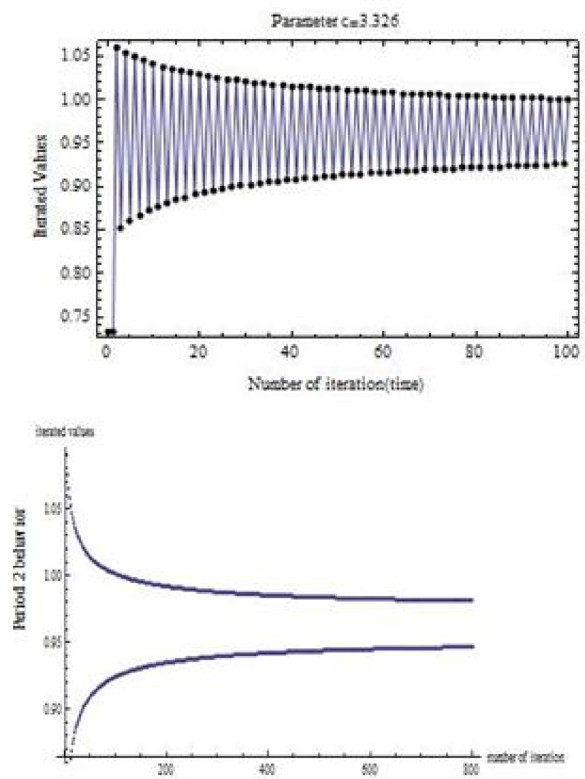
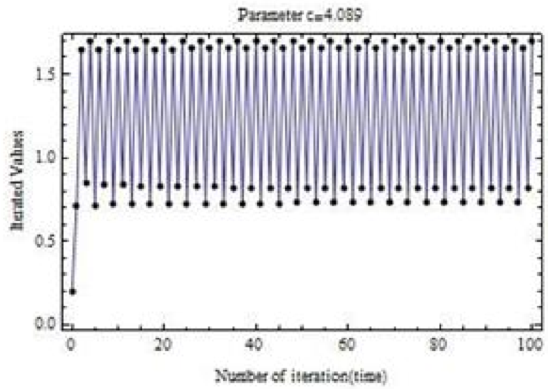


Fig. 5: Time series plot of x_n vs. n at c=3.28



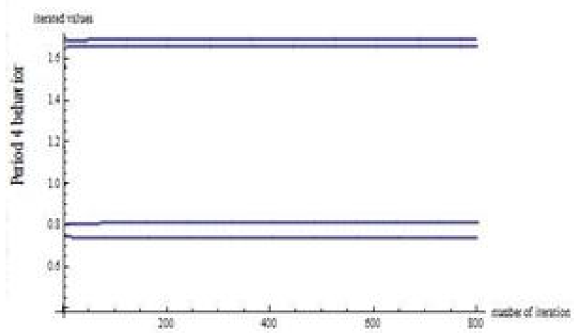
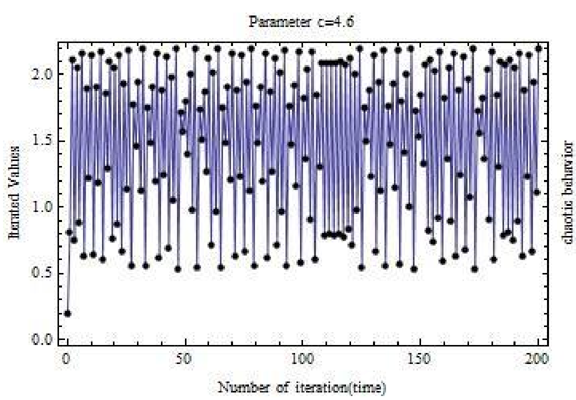


Fig. 5(a): Time series plot of x_n vs. n at c=3.28

For c just greater than 3.326 I,e when c = 4.086 the curve oscillates between four fixed points showing period four behavior.



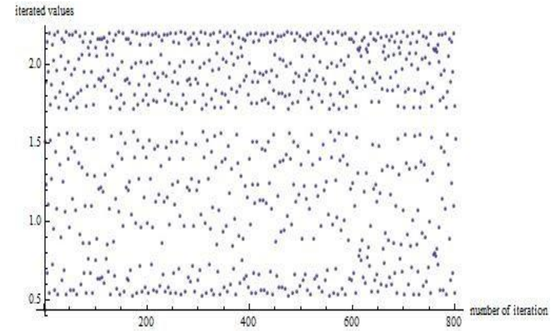
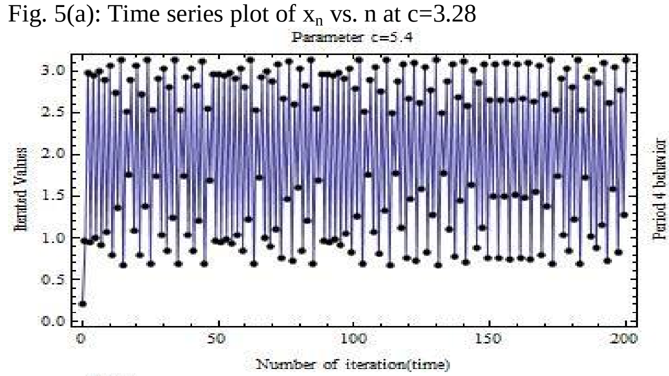


Fig. 5(a): Time series plot of x_n vs. n at c=3.28



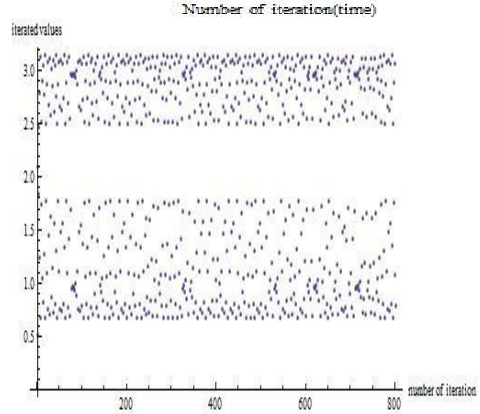
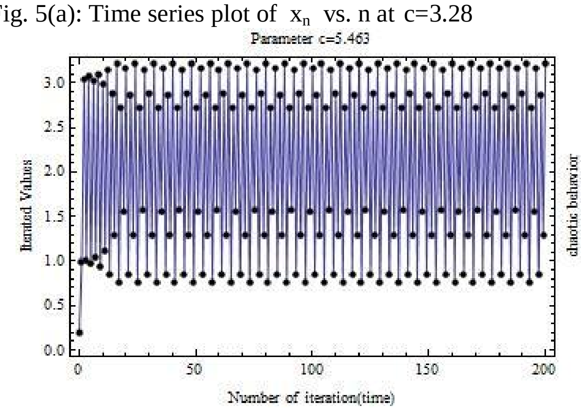


Fig. 5(a): Time series plot of x_n vs. n at c=3.28



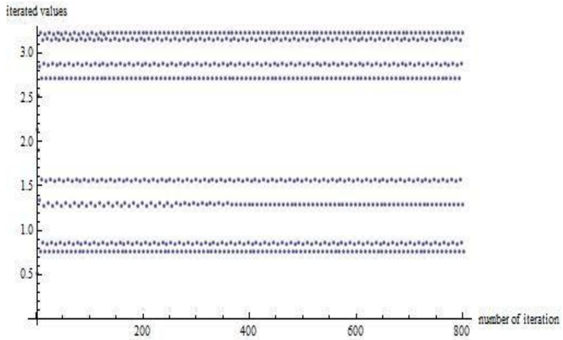
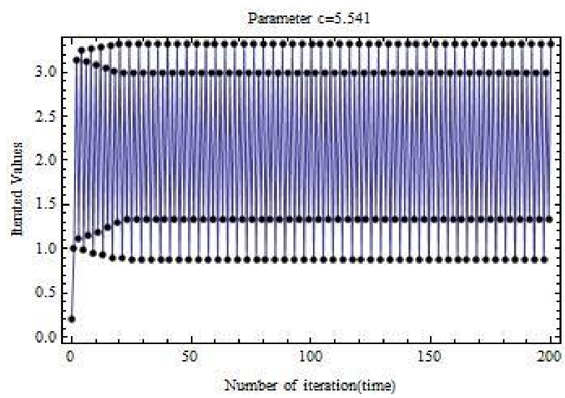


Fig. 5(a): Time series plot of x_n vs. n at c=3.28



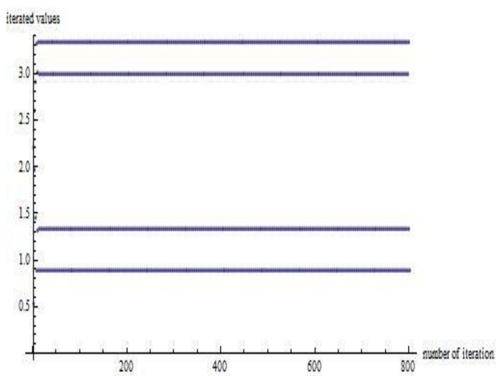
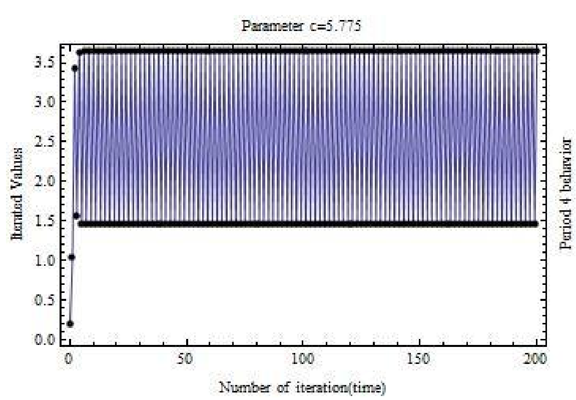


Fig. 5(a): Time series plot of x_n vs. n at c=3.28



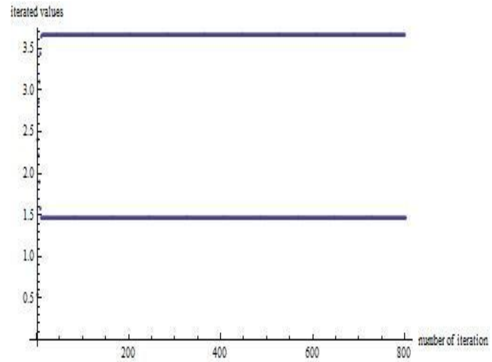
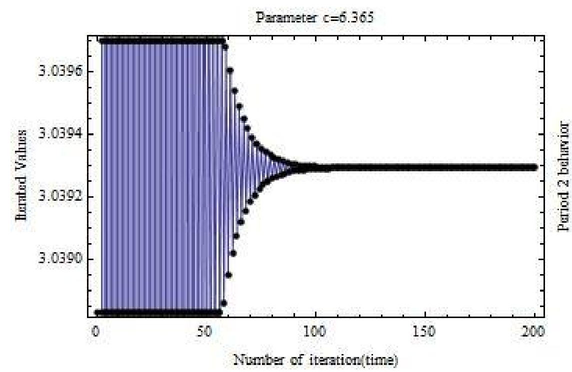


Fig. 5(a): Time series plot of x_n vs. n at c=3.28



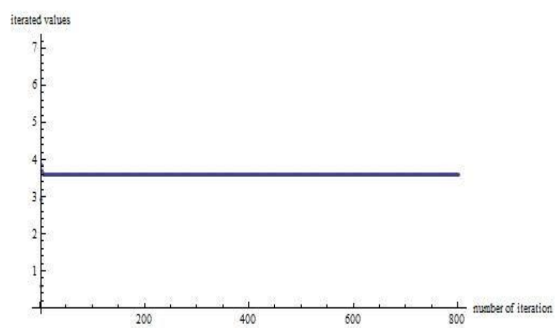


Fig. 5(a): Time series plot of x_n vs. n at c=3.28

Lyapunov Exponent:

Lyapunov Exponent is a method to determine whether or not a system is chaotic. It is the measure of divergence of two trajectories starting very close to each other. Lypunov exponent may be computed for a sample of points near the attractor to obtain the average Lypunov Exponent. If at least one of the Lypunov Exponent is positive than the system is chaotic, if negative then the orbit is periodic and when it is zero , a bifurcation occurs.

Let us consider a map

$$x_n = f(x_{n-1})$$

Let the initial condition be x_O , consider a nearby point $x_O + \delta_O$,where the initial separation δ_O is extremely small, [2,4,6,11].

Then we have

$$|\delta O| e^{n\lambda(x_o)} = | f^n(x_0 + \delta_o) - f^n(x_0) |$$

i,e, $e^{n\lambda(x_0)} = \left| \frac{f^n(x_0+\delta_c) - f^n(x_0)}{\delta_o} \right|$

where λ is the Lyapunov exponent. A positive Lyapunov Exponent is a signature of chaos. By taking $\delta O \to 0$ and applying chain rule of differentiation we get

λ (x0)$=\frac{1}{n}$

In

(| $f'(x_0)$|| $f'(x_1)$|……… |$f'(x_{n-1})$|)

$=\frac{1}{n}$

(In | $f'(x_0)$|$+In$| $f'(x_1)$|+………$+In$ |$f'(x_{n-1})$|)

λ (x0) $=\frac{1}{n}\sum_{i=0}^{n-1} In\ |f'(x_i)|$

Taking the limit n → ∞, the expression for Lyapunov exponent λ(x0) is

$$\lambda\text{ (x0)}= \lim_{n\to\infty} \frac{1}{n} \log\left|\frac{df^n(x_0)}{dx}\right|$$

Now if λ < 0, then $\lim_{n\to\infty} f^n(x_0)$ = $\lim_{n\to\infty} f^n(x_0+\delta)$

that is, the system is not sensitive to the initial value.

Again if h >0 then

$$|f\text{n}(xO + \delta) - f\text{n}(xO)|$$

diverges to infinity with exponential growth which corresponds to chaos.

In the cubic map

$$f[x] = 0.35x3 - 2.75x2 + cx$$

period doubling starts at c=3.3286…. and ends at the accumulation point

$$c_\infty = 4.339307529$$

which is the point corresponding to an orbit of period 2œ. The Lypunov Exponent at the point of accumulation is not positive thus the corresponding attractor is periodic. For values c >c_∞ ,the lyapunov exponent is positive reflecting the chaotic behavior of the corresponding attractors. Again for c < c_∞, the Lyapunov exponent is negative except at the bifurcation points. At these points the Lyapunov exponents are found to be 0. From the fig it can be seen that the chaotic region remains till c=5.463 .

Thereafter Lypunov exponent becomes negative showing periodic behavior and

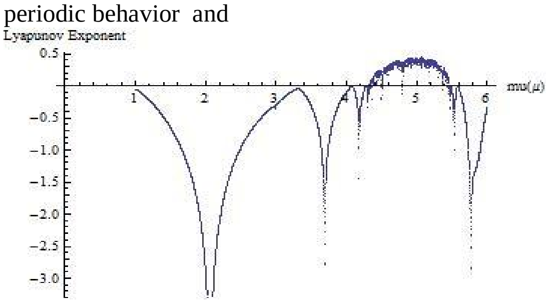


Fig:6 graph of Lyapunov Exponent for the parameter from 0 to 6.3

The following sequence shows how the Lyapunov exponent changes along a trajectory. The purpose is to test the stability of an orbit.

Parameter c	Lyapunov exponent
3.28	-0.03107
3.326	0.0
4.086	0.0
4.3257	0.0
4.3367	0.0
4.412	0.0
4.6	0.283033
4.9	0.369165
5.463	-0.00144
5.541	-0.49225
5.775	-2.75729
6.365	-1.69057

CONCLUSION:

In this paper we have seen that simple cubic map exhibit periodic doubling route to chaos and thereby simultaneously showing undoubling phenomenon. Also we have seen that our map successfully converges to Feigenbaum universal constant (delta). Bifurcation graph shows that the chaotic ……..region occurs at accumulation point 4.339308…. which is can be observed till 5.4…After that the chaotic region is again followed by undoubling of the bifurcation. Time series plot shows how the iterated values changes to chaotic ones. We see from the graph how Lyapunov Exponent changes its sign from negative to positive as the control parameter varies. This negative value indicates about regular behavior of the periodic points and positive values gives the signal of chaos.

Open Problems: (1) How can we determine various fractal dimensions, viz, Housdorff dimension, Information dimension, Topological dimension , Box dimension?

(2) Can we develop a sophisticated method in order to control chaos of all chaotic cubic maps ?

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