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PROPERTIES ON BIFURCATION AND TOPOLOGICAL ENTROPY IN SYMBOLIC DYNAMICS

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Abstract: The prime objectives of this paper are to establish Feigenbaum bifurcation universality on the population model :
$f_p(x) = x + px(1-x)$
where $x \in [0, 4/3]$ and $p \in]0, 3]$ is a positive parameter; and to obtain topological entropy on unimodal map: f(x) = λx(1-x), x ∈ [0,1] , λ ∈ (1,4]

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1.0 INTRODUCTION

1.01 Feigenbaum Theory:

Chaos theory began at the end of nineteen century (around 1890) with some great initial ideas, concepts and results of the famous French mathematician, Henri Poincare. Also the recent path of the theory has many fascinating successful stories. Probably, the most beautiful and important one is the route from order into chaos, i.e., the Feigenbaum university. Mitchell J. Feigenbaum, a renowned American particle theorist is known as the founder of the period-doubling bifurcation that may be described as a universal route to chaos – an exciting discovery in nonlinear dynamical system. Many new universal properties have been discovered by Feigenbaum for families of maps which depend on a parameter λ. One of his fascinating discoveries is that if a family 'f' represents period- doubling bifurcation then there is an infinite sequence $\{\lambda_n\}$ of bifurcation values such that

$$\delta = \lim_{n\to\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n}$$

where δ is a universal number known as the Feigenbaum constant, which does not depend at all on the form of the specific family of maps. The value of δ is 4.6692016091029….. in the dissipative case and 8.72109720…. in the conservative case. Furthermore, his observation suggests that there is a universal size scaling in the period- doubling sequence designated as the Feigenbaum α - value

$$\alpha = \lim_{n\to\infty} \frac{d_n}{d_{n+1}} = 2.5029..........$$

where d_n is the 'size' of the bifurcation pattern of period 2^n just before it gives birth to period 2^{n+1} .

Bifurcation theory is a method for studying how solutions of a nonlinear problem and their stability changes as the parameter varies .The onset of chaos is often studied by bifurcation theory. For example, in certain parametrized families of one dimensional maps, chaos occurs by infinitely many period- doubling (P-D) bifurcations . In the case of a diffeomorphism f, P-D bifurcations (or Flip bifurcations or Sub harmonic bifurcations) occur when one of the eigen values of the derivative Df (x) equals -1, [1,3,6,10, 13].

symbolic dynamics is the practice of modeling a topological or smooth dynamical system by a discrete space consisting of infinite sequences of abstract symbols, each of which corresponds to a state of the system, with the dynamics (evolution) given by the shift operator. Formally, a Markov partition is used to provide a finite cover for the smooth system; each set of the cover is associated with a single symbol, and the sequences of symbols result as a trajectory of the system moves from one covering set to another, [2,4,9,12].

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1.02Topological Dynamical System: A dynamical system is a particular type of function used to model time-varying processes. Further, a topological dynamical system is a pair (X, f) where the phase space X is a compact or a locally

compact metric space and $f : X \rightarrow X$ is a continuous transformation or homeomorphism.

That is, topological dynamics concerns itself with groups of homeomorphisms or semi groups of continuous maps of phase spaces.

1.03 Shifts: Two-sided and One-sided :

Shifts are basically viewed as digital coding of information of different nature. The problem of encoding in various practical situations has been simplified through shifts. Shift space, compact and shift invariant subsets of the full shift, are mainly used to model smooth dynamical systems through Markov Partitions.

Let A be a finite set. The set of all infinite two-sided sequences of symbols from the finite set A is denoted by A^Z and is known as the two-sided or bi-sided full A-shift or simply the full A-shift. In this case, the finite set A and its elements are referred to as the alphabet and letters respectively. Generally, A contains typical symbols like 0, 1, 2, 3…. or a, b, c, d, etc. The full shift over the alphabet A = {0, 1, 2, …., m-1} is termed as the full m-shift and it is generally denoted by Σ_m or $X_{[m]}$. A typical point x in a shift denoted as

$x = \ldots\ldots x_{-3}x_{-2}x_{-1} \cdot x_0x_1x_2x_3 \ldots\ldots$ where $x_i^{\prime s}$ $\in$ alphabet

A finite sequence of symbols from the alphabet A of the type $x_{\lambda_1}x_{\lambda_2}x_{\lambda_3}\ldots x_{\lambda_k}$ is called a word or a block of length k or simply a k-block over A. For i, j $(> i) \in Z$, $x_{[i,j]}$denotes the block $x_ix_{i+1}x_{i+2}\ldots x_j$ of the typical point $x = \ldots\ldots\ldots x_{-3}x_{-2}x_{-1} \cdot x_0x_1x_2x_3 \ldots\ldots$ from the position i to the position j. Further, the block $x_{[-k, k]} = x_{-k}x_{-k+1} \ldots\ldots x_k$, $k \in N$, is generally known as the central (2k + 1) – block of x and the role of the central block of points are very essential in studying the dynamics of the full shifts as well as other shift spaces. Symbolic representation of invertible maps gives rise to bi-sided shifts. On the other hand, if $m(\geq 2) \in N$ and A = {0, 1, 2, 3, ……., $m - 1$}, then A^m denotes the set of all one-sided right-infinite sequences of symbols in A. It is also denoted by the symbol Σ_m^+or simply by the symbol Σ_m. That is, $\Sigma_m^+ = A^N = \{0,1,2,3....,m-1\}^N$ $=\{(x_i)_{i=1}^{\infty} : x_i^{\prime s} \in \{0,1,2,3,....,m-1\}\}$. $\Sigma_m^+ = A^N$is known as the one-sided full A-shift. Symbolic representation of non-invertible maps gives rise to one-sided shifts. Restricting finite number of words as in bi-sided shifts, we also get one-sided shift spaces.

1.04: Topological Partition:

Let (M, ϕ) be a topological dynamical system. A topological partition of M is a finite collection $P = \{M_0, M_1...., M_{m-1}\}$ of disjoint non-empty open sets of M such that $M = \bigcup_{i=0}^{m-1} \overline{M_i}$, i.e., $\overline{P} = \{\overline{M_0}, \overline{M_1}, \overline{M_2}, ..., \overline{M_{m-1}}\}$ covers M . By virtue of this partition, we get a symbolic representation of the TDS (M, ϕ) in the alphabet A = (0, 1, 2,….., m - 1}, If $L_{P,\phi}$,denotes the collection of all the allowed words of the form $w = a_1a_2a_3....a_n$ for P, ϕ, then it is the language of a unique shift space which is denoted by $X_{P,\phi}$. This shift space $X_{P,\phi}$ is the symbolic dynamical system corresponding to P, ϕ and represents the system (M, ϕ). It can be proved that $w = a_1a_2a_3....a_n$is allowed for P, ϕ if $\bigcap_{j=1}^{n} \phi^{-1}(P_{a_j}) = \phi$. If ϕ is not necessarily invertible, then the one-sided shift space $X_{P,\phi}^+$ is the one-sided symbolic dynamical system corresponding to P, ϕ.

1.05 Markov Partition

Let (M, ϕ) be an invertible dynamical system and P = $\{M_0, M_1, \ldots, M_{m-1}\}$ be a topological partition of M . Then P gives a symbolic representation of the invertible dynamical system (M, ϕ) if for each point$a \in X_{P,\phi}$,the intersection $\bigcap_{n=0}^{\infty} \overline{D_n(a)} = \bigcap_{n=0}^{\infty} \overline{\left[\bigcap_{k=-n}^{k=n} \phi^{-k}(P_{a_k})(a) \right]}$ consists of exactly one point, If P gives a symbolic representation of (M, ϕ) and $X_{P,\phi}$ is a shift of finite type, the topological partition P is called a Markov partition for (M, ϕ).

For non-invertible dynamical systems, replacing $D_n(a)$ with $D_n^+(a)$, where $D_n^+(a) = \bigcap_{k=0}^{n} \phi^{-k}(P_{a_k})(a)$ and $X_{P,\phi}$ with $X_{P,\phi}^+$, we have a one-sided Markov partition.

1.06 Symbolic dynamics: Let Q be a topological partition. Suppose $f : X \rightarrow X$ is monotone continuous on each element of L.

The lower symbolic dynamics $\underline{\Sigma}(f, Q)$ of f is the set of sequences $\vec{s}$ such that$\forall m \in N$, there exists $\in > 0$ $such that for all \epsilon$-perturbations $f\epsilon$of f with the same monotone branches, $f\epsilon$has an orbit whose Q-itinerary has the same m-prefix of $\vec{s}$.

- The upper symbolic dynamics $\overline{\Sigma}(f,Q)$ of f is the set of all sequences $\vec{s}$ such that $\vec{s}$ for all $\epsilon > 0$, there exists an ϵ-perturbation $f\epsilon$ of f with the same monotone branches such that $f\epsilon$ has an orbit with Q-itinerary $\vec{s}$.

1.07 Topological entropy:

The topological entropy of a map $f : X \to X$ is defined as:

$$h(f) = \lim_{\in\to 0}\left(\limsup_{n\to\infty} \frac{1}{n} \log N(n,\in) \right)$$

where $N(n,\in) = N(n,\in) = \max_{S\in V_{n,\in}} |S|$ and

$$V_{n,\in} = \{S \subset X \mid \forall x, y \in S \max\{d(f^i(x), f^i(y)) : 0 \le i < n\} \ge \in\}$$

1.08 Kneading Theory:

Let f be a multimodal map defined on the interval I = [a, b] with critical points $a < c_1, \ldots, c_{l-1} < b$. Let $I_J = [c_{j-1}, c_j]$ for $j = 1, \ldots, l$, where we take $c_0 = a$ and $c_l = b$. For let ε_j be the sign of f over the interval I_j. For a sequence $\vec{s} \in \{1, \ldots\ldots, l\}^N$, denoted by $\varepsilon_{\vec{s},j}$ or $\varepsilon_{s_0,s_1,\ldots,s_{j-1}}$ the product $\varepsilon_{s_0}\varepsilon_{s_1}\ldots\varepsilon_{s_{j-1}}$.

We define an ordering on the itineraries of f. Suppose $\vec{s}$ and $\vec{t}$ are sequences, and $j = \min\{i \mid s_i \ne t_i\}$, and suppose $s_j < t_j$. Then $\vec{s} < \vec{t}$ iff $\in\varepsilon s_j < \in\varepsilon t_j$, where $\in = \varepsilon_{\vec{s},j} = \varepsilon_{\vec{t},j}$.

We now consider whether there exists an orbit with a given itinerary. Suppose the itineraries of the images of the critical points $c_1, \ldots, c_{l-1}$ *are* k_j, *that is* $\vec{k}_j = \iota(f(c_j))$. Suppose $\vec{t}$ is the itinerary of a point x. Then if $c_i < x < c_{i+1}j$, then either $f(c_i) < f(x) < f(c_{i+1})$ *or* $f(c_{i+i}) < f(x) < f(c_i)$ depending on the sign of ε_{i+1}. Since $\iota(f(x))$ is $\sigma(\iota(x))$, where $\iota(x)$ is the itinerary of x, We deduce that $\vec{t}$ is an itinerary and only if, $\sigma^{n+1}(\vec{t}) \in [\vec{k}_{t_n}, \vec{k}_{t_{n+1}}]$, where the endpoints of the interval may be reversed. We may also consider the endpoints of the intervals.

1.09 Computing Symbolic Dynamics using Covering Relations:

We will consider the computation of symbolic dynamics for a P-continuous map f with respect to the partition Q. We let B be the boundary points of Q, C the critical points of f and D the discontinuity points of f.

Our basic strategy for computing symbolic dynamics is as follows

Algorithm Scheme : Compute the topological partition L such that f is monotone and continuous on each piece of L.

1. Refine the partition $L \vee Q$ to obtain a partition R.
2. on the refined partition R, compute the symbolic dynamics by considering covering relations $f(R) \supset R', f(R) \subset R'$ and $f(R) \cap R' \ne \emptyset$.

1.10 Covering relations:

The simplest way to extract symbolic dynamics is directly via covering relations. Given a partition R and covering relations between its elements we want to construct sofic shifts under- and over-approximating the symbolic dynamics.

Let $f : X \to X$ be a P-continuous map and Q a partition of X. Let R be a partition of X refining Q and P. Let R be the partition L into monotone continuous pieces of f.

Define covering relations $R \to R'$ and $R - \to R'$ on $R \times R$ by $R \to R'$ if $f(R) \supset R'$ and $R \to R'$ iff(R) $\cap R' \ne \Phi$

Armed with these concepts and by using MATHEMATICA, we now proceed for our principal results.

2.00 The Main Results:

This section is primarily concerned with the birth and flowering of the beautiful phenomena of period-doubling bifurcations leading to chaos and subsequent achievement of Feiegenbaum universality on the **Verhulst population model:**

$$f_p(x) = x + px(1-x)$$

where $x \in [0, 4/3]$ and $p \in]0, 3]$ is a positive parameter.

To find points of Period-one, it is necessary to solve the equation given by

$$f_p(x) = x + px(1-x) = x$$

which gives the points that satisfy the condition $x_{n+1} = x_n$ for all n. The solutions are $x_1^* = 0$ and $x_2^* = 1$. These two fixed points are the intersection of the graphs of $y = f_p(x)$ and $y = x$. The periodic points x_1^* and x_2^* are shown in the figure 1.1. The stability of the critical points may be determined using the following theorem:

Theorem: suppose the map $f_p(x)$ has a fixed point at x^*. Then the fixed point is stable if

$$\left| \frac{d}{dx} f_p(x^*) \right| < 1$$

and unstable if

$$\left|\frac{d}{dx}f_p(x^*)\right| > 1$$

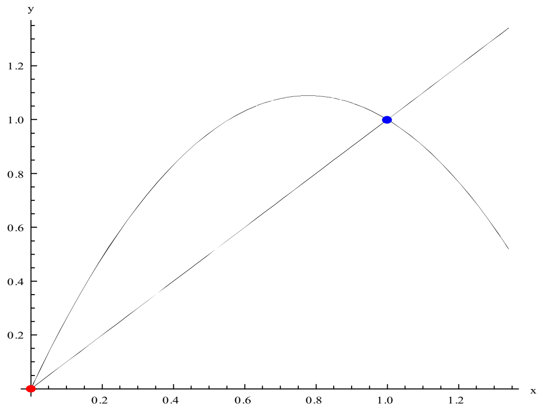

Fig 1.1 Graphs of y= f_p(x) and y= x for p= 1.8

Using the above theorem, we have $\left|\frac{d}{dx}f_p(x_1^*)\right| = 1+p$.

Thus the fixed point $x_1^* = 0$ is always unstable as $p > 0$. Again $\left|\frac{d}{dx}f_p(x_2^*)\right| = 1-p$, the point $x_2^* = 1$ stable for $0 < p < 2$ .Having studied the dynamics of the quadratic iterator f_p in detail for parameter values between *0* and *2*, we continue to increase *p* beyond *2*. For such large parameter values the fixed point x_2^* is not stable anymore, it is repellor. Hence, the **first bifurcation** value is $p_1 = 2$.

To find points of period-two, we consider the iterated map $f_p^2(x)$. Here,

$$f_p^2(x) = x + p(1-x)x + p(1-x-p(1-x)x)(x+p(1-x)x)$$

The periodic points of $f_p^2(x)$ are given by the equation

$$f_p^2(x) = x \tag{2.1}$$

Here, x_1^* and x_2^* are two solutions of (2.1), since points of period-one repeat on every second iterate. So, the equation factorizes as follows:

$$x(x-1)(-p^3x^2 + p^3x + 2p^2x - p^2 - 2p) = 0$$

The equation $-p^3x^2 + p^3x + 2p^2x - p^2 - 2p = 0$ has roots at

$$x_{11}^* = \frac{(2+p)+\sqrt{p^2-4}}{2p}$$ and

$$x_{12}^* = \frac{(2+p)-\sqrt{p^2-4}}{2p}$$

These four points are the intersection of the graphs of $y = f_p^2(x)$ and *y=x*, see figure1.*2*. The periodic points x_1^*, x_2^*, x_{11}^* and x_{12}^* marked as *A1, A2, B1* and *B2* in the figure 1.2

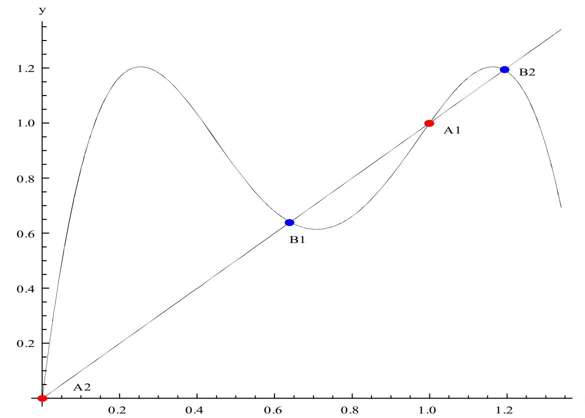

Fig 1.2 Graphs of $y = f_p^2(x)$ and y= x for p= 2.4

Stability of the first two fixed points is already discussed. We now discuss the stability of the new points: x_{11}^* and x_{12}^*. Considering only parameters between 0 and 3, we note that these new solutions are defined only for $p \geq 2$. Moreover, at $p = 2$, we get $x_{11}^* = x_{12}^* = \frac{2+p}{2p}$, i.e. these two solutions bifurcate from the fixed point x_2^*. Thus, at parameter value $p = 2$, our map orbits undergo period-doubling bifurcations. Just below $p = 2$ the orbits converge to a single value of *x*. Just above $p = 2$, the orbits tend to this alteration between two values of *x*.

These two points form a two-cycle, one being the image of the other. Let us see how the derivatives of the map function $f_p(x)$ and of the second iterate function $f_p^2(x)$ change at the bifurcation value. The equation:

$$\left.\frac{df_p(x)}{dx}\right|_{x=x_2^*} = 1-p$$ tells us that function $\frac{df_p(x)}{dx}$ passes through the value *-1* as *p* increases through *2*. Next we can evaluate the derivative of the second iterate function by using the chain-rule of differentiation:

$$\frac{df_p^2(x)}{dx} = \frac{d}{dx}[f_p(f_p(x))] = \left.\frac{df_p}{dx}\right|_{f_p(x)} \left.\frac{df_p}{dx}\right|_x$$

If we now evaluate the derivative at one of the above two new fixed points, say x_{11}^* then we find

$$\left.\frac{df_p^2(x)}{dx}\right|_{x_{11}^*} = \left.\frac{df_p}{dx}\right|_{x_{12}^*} \left.\frac{df_p}{dx}\right|_{x_{11}^*} = \left.\frac{df_p^2(x)}{dx}\right|_{x_{12}^*} \tag{2.2}$$

In arriving at the last result, we made use of $x_{12}^{*} = f_p(x_{11}^{*})$ for the two fixed points. The derivative of $f_p^2(x)$ are the same at both the fixed points that are actually part of the two-cycle. This result implies that both of these fixed points are either attracting or both are repelling, and that they have the same 'degree' of stability or instability. Again, since the derivative of $f_p(x)$ equals *-1* for the parameter $p = 2$, equation 2.3 tells us that the derivative of $f_p^2(x)$ equals +1 for $p = 2$. As *p* increases further, the derivative of $f_p^2(x)$ decreases and the fixed points become stable. Besides, the unstable fixed point of $f_p(x)$ located at x_2^* is also an unstable fixed point of $f_p^2(x)$. Fig 1.2 shows the graph of $f_p^2(x)$ for a value of *p* just above parameter value *p = 2.13*. For *p*, just greater than *2*, we see that the slope of $f_p^2(x)$ at the fixed point x_{11}^* and x_{12}^* is less than one, and hence they are stable fixed points of $f_p^2(x)$.

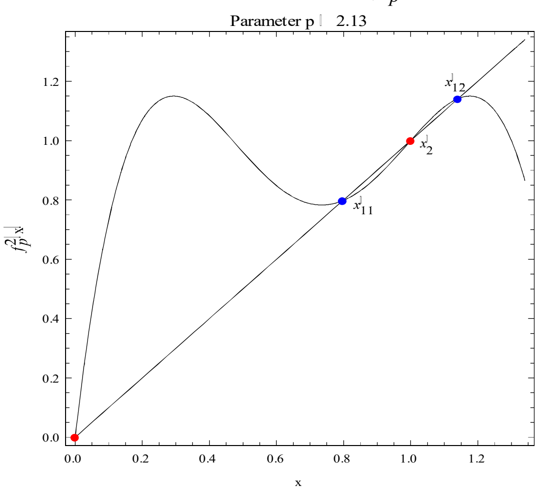

Fig 1.3 Graph of $f_p^2(x)$

The 2-cycle fixed points of $f_p^2(x)$ continue to be stable fixed points until $p_2 = 2.449489742783...$ We have values of x_{11}^* and x_{12}^* as *0.619573155869….* and *1.196923425058….* respectively at parameter value $p_2 = 2.449489742783...$. Also for this value of *p*

$$\left.\frac{df_p^2(x)}{dx}\right|_{x=0.619573155869...} = -1, \quad \left.\frac{df_p^2(x)}{dx}\right|_{x=1.196923425058......} = -1$$

The above results guarantee that if a system is stable or unstable at a periodic point, then the system is so at any other periodic point. So our study will be complete if we study the dynamics at any of the periodic points.

We can find that for values of *p* larger than p_2, the derivative is more negative than *-1*. Hence for *p* values greater than p_2, the 2-cycle points are repelling fixed points. We find that for values just greater than p_2, the orbits settle into a 4-cycle, that is, the orbit cycles among *4* values which we can label as $x_{21}^*, x_{22}^*, x_{23}^*, x_{24}^*$.

These points are the intersection of the graphs of $y = f_p^4(x)$ and $y = x$ in the above fig.1.3. To determine these periodic points analytically, we need to solve an eight degree equation, namely $f_p^4(x) = x$ which is manually cumbersome and time consuming. Therefore, for finding periodic points, bifurcation values of f_p^4 as well as for higher iterated map functions, we have to write a computer program. We write here a C-program for our purpose. Of course, with the help of **MATHEMATICA**, for the parameter value $p = 2.5$ the approximate values of the fixed points $x_{21}^*, x_{22}^*, x_{23}^*, x_{25}^*$ are calculated as *0.537(A), 0.703(B), 1.159(C) 1.227(D)* respectively.

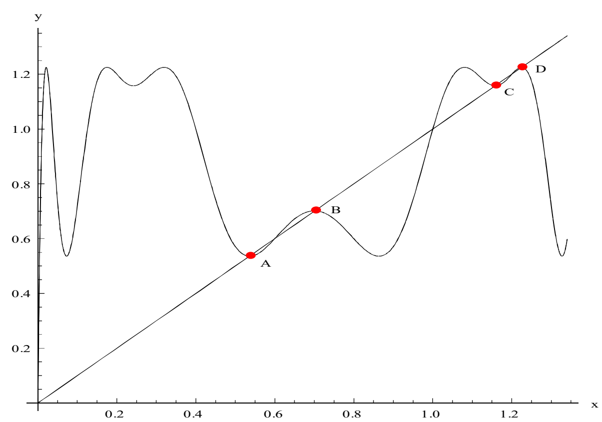

Fig 1.4 Graphs of $y = f_p^4(x)$ and y= x for p= 2.5

Then using the relation (2.3), an approximate value p_3' of p_3 is obtained. Since the Secant method needs two initial values, we use p_3 and a slightly larger value, say, $p_3 + 10^{-4}$ as the two initial values to apply this method and ultimately obtain p_3. In like manner, the same procedure is employed to obtain the successive bifurcation values $p_4, p_5....$ etc. to our requirement. Through our numerical mechanism, we obtain some periodic points and bifurcation values. In Table 1.1 the first nine bifurcation points together with their corresponding periodic points are shown:

Table2.1

One of Periodic points	Bifurcation values
$x_1 = 1$	$p_1 = 2.00$
$x_2 = 0.619573155869....$	$p_2 = 2.449489742783.....$

$x_3 = 0.506088071014...$	$p_3 = 2.544090359552.....$
$x_4 = 0.481986116641...$	$p_4 = 2.564407266095....$
$x_5 = 0.477095904017...$	$p_5 = 2.568759419544....$
$x_6 = 0.477109948703...$	$p_6 = 2.569691609801....$
$x_7 = 0.475897082...$	$p_7 = 2.569891259378....$
$x_8 = 0.4771477219...$	$p_8 = 2.569934018374.....$
$x_9 = 0.476002501444...$	$p_9 = 2.569943176048....$

(Here we denote the value of p for the k th bifurcation by p_k)

Based on these values, the ratios of successive separations of bifurcation points are given by,

$$\frac{p_2 - p_1}{p_3 - p_2} \approx \frac{p_3 - p_2}{p_4 - p_3} \approx \frac{p_4 - p_3}{p_5 - p_4} \approx \frac{p_5 - p_4}{p_6 - p_5} \approx \frac{p_6 - p_5}{p_7 - p_6} \approx ...$$

and have a particular scaling associated with them. We see that

$$\delta_1 = \frac{p_2 - p_1}{p_3 - p_2} = 4.7514..., \delta_2 = \frac{p_3 - p_2}{p_4 - p_3} = 4.6562...,$$

$$\delta_3 = \frac{p_4 - p_3}{p_5 - p_4} = 4.6682...$$

$$\delta_4 = \frac{p_5 - p_4}{p_6 - p_5} = 4.6687..., \delta_5 = 4.6691..., \text{and so on.}$$

The ratios tend to a constant as k tends to infinity: more formally

$$\lim_{k \to \infty} \left[\frac{p_k - p_{k-1}}{p_{k+1} - p_k} \right] = \delta = 4.669201...$$

The nature of δ is universal i.e. it is the same for a wide range of different iterators

2.1 The unimodal map:

Consider the unimodal map: $f_1(x) = \lambda x(1-x)$ with $\lambda =$ 3.92 Let a = 0 and b = 1 be the endpoints of the interval I and c = 0.5 be the critical point. Let $I_0 = [a, c]$ and $I_1 = [c, b]$. The orbit of the critical point is given by $f^i(c) = c_i$ with

$$c_0 = 0.5, c_1 = 0.98, c_2 \approx 0.077, c_3 \approx 0.278, c_4 \approx 0.787$$

$$c_5 \approx 0.657, c_6 \approx 0.883, c_7 \approx 0.405, c_8 \approx 0.945, c_9 \approx 0.204$$

The itinerary of $f_1(c)$ is therefore itin $(f_1(c))$ = 100111010……

Consider the partition into six intervals with boundary points

$p_0 = a, p_1 = c_2, p_3 = c_3, p_4 = c_0, p_5 = c_4, p_6 = c_1, p_7 =$ b.

Note that $f(p_5) \in (P_4, p_5)$. Taking $J_i = [p_i, p_{i+1}]$ we have

$$f_1(J_1) \supset J_2 \cup J_3; f_1(J_2) \supset J_4; f_1(J_3) \supset J_4; f_1(J_4) \supset J_1 \cup$$

and also $f_1(J_3) \cap J_3 \neq \emptyset$ and $f(J_4) \cap J_3 \neq \emptyset$.

Now consider the computation of entropy. Using the lower and upper symbolic dynamics on the partition, we obtain entropies of 0.41962 and 0.73286. The upper shift on the refined partition has multiple orbits with the same itinerary on I_0 and I_1. If we consider the entropy of the shift itself (by using additional states J_{12} and J_{34}) we obtain 0.583 .

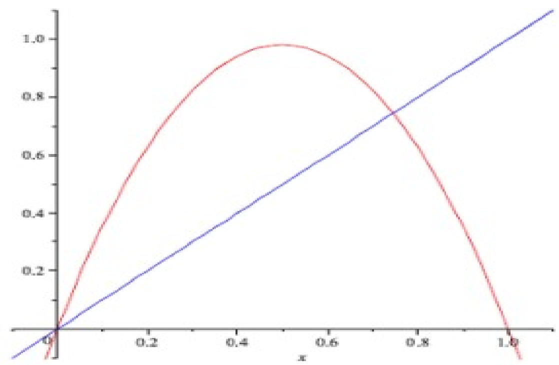

Figure 1.5 Unimodal map

Steps	Entropy
1	[0:0.69314]
2	[0.48121:0.69315]
3	[0.48121:0.60938]
4	[0.54353:0.60938]
5	[0.54353:0.58356]
6	[0.54353:0.56240]
7	0.54761:0.56240]
8	[0.55642:0.56240]
9	[0.55643:0.56099]
10	[0.55842:0.56099]
11	[0.55990:0.56099]
12	[0.55990:0.56073]
13	0.55990:0.56026]
14	[0.55998:0.56026]
15	[0.56014:0.56026]

Table 2.2: Entropy of the unimodal map

Improvement of entropy using kneading theory:

If we use the kneading theory, we see that by setting $f_1(c_4) = c_4$ we have itin $(f_1(c)) = 1001111$ …… which is higher in the unimodel order, and setting $f_1(c_4) = c_0$ we have itin$(f(c)) = 1001_1^0100$….. which is lower than 10011101… . The corresponding lower and upper entropies are 0.54354 and 0.571.

Consider a unimodal map with kneading invariant $\overline{1001_1^0}$, so c is periodic and $f_1^5(c) = c$. The images of c are ordered $a < f_1^2(c) < f_1^3(c) < c < f_1^4(c) < f_1(c) < b.$ We can compute the topological entropy of the shift, and obtain a value of 0.57058.

Now consider a unimodal map with kneading invariant $\vec{k} = 10011....$. From the kneading theory, we know $10011 \succ 1001_1^0$ so we have $h_{top}(f_1) \geq 0.64$. Define $R_1 = [f_1^2(c), f_1^3(c)]$, $R_1 = [f_1{}^3(c), c]$, $R_3 = [c, f_1{}^4(c)]$ and $R_4 = [f_1^4(c), f_1(c)]$. When using the forward refinement strategy, since $f_1^5(c) \in R_3$, the interval R_3 maps to $[f_1^5(c), f_1(c)]$ and the interval $[f^4(c), f(c)]$ maps to $[f_1^2(c), f_1^5(c)]$ which together cover R_3, but neither does individually. Hence neither of the transitions $R_3 \to R_3$ nor $R_4 \to R_3$ in the automaton for the lower symbolic dynamics. The entropy bound obtained drops to 0.41962.

The overall effect of the kneading theory is to "choose" a transition, either $R_3 \to R_3$ or $R_4 \to R_3$, to put in the lower symbolic dynamics, while still ensuring that the dynamics is a lower bound. The chosen transition is the one giving least entropy.

Steps	Entropy	Running time
1	[0:0.7093147180559]	0.01
2	[0:0.7023147180559]	0.04
3	[0.491213225059:0.6931471805599]	0.06
4	[0.4932211825059:0.6093778634360]	0.07
5	[0.5382135072497:0.5705796667792]	0.09
6	[0.541235072497:0.5705796667792]	0.10
7	[0.549794599694:0.5623991486459]	0.10
8	[0.55231930430:0.5623991486459]	0.10
9	[0.55567519816:0.5623991486459]	0.12
10	[0.56014256097:0.560988810813]	0.15
15	[0.56022219753:0.560259207813]	0.19
20	[0.56023564272:0.560235821005]	0.26
25	[0.5602357435170:0.560235669923]	0.28
30	[0.56023589560:0.560235638765]	0.29
35	[0.56023597749:0.560235636490]	0.39
40	[0.560236846370:0.560235636375]	0.45

Table 2.3: Entropy of the unimodal map with kneading algorithm

CONCLUSION

The study of chaos in symbolic dynamical models is quite interesting. Although there are so many methods for finding bifurcation values, we have developed own numerical mechanism for establishing Feigenbaum tree of bifurcation values leading to chaotic region the study of which is intrinsically marvelous. Our method seems to be applicable to all the chaotic models. Topological entropy is the measure of chaos. Here our results indicate the existence of chaos in our nonlinear model.

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