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SOLVING THE (0 -1) KNAPSACK PROBLEM BY AN ADAPTED TRANSPORTATION ALGORITHM

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Abstract* - In this paper we link the zero-one knapsack problem to the linear transportation problem then solve it by using an adaptation of the transportation algorithm. The Vogel Approximation Method is applied to find an initial solution. It consists in assigning to each row and column a penalty which is the difference between the two least costs. The largest penalty indicates the line to be allocated first. Then the variable with the least cost on that line is assigned. For the zero-one knapsack problem, the Vogel method is shown to be equivalent to the Greedy Algorithm. That initial solution is then improved by using the dual variable and resulting reduced cost. We detect conditions which indicate that the optimal solution is reached. We also prove that when no further cost's reduction is possible, then an optimal solution is obtained. This approach opens a new field of research which treats the zero-one knapsack problem as a transportation problem.

Keywords. Knapsack problem, Linear Transportation problem, Assignment problem, Vogel Approximation Method, Vogel Penalties, Adapted Transportation Algorithm**.**

1. INTRODUCTION

Given a set of n items $j = 1, \dots, n$, each having a weight w_j and inducing a unit profit P_j, the knapsack problem (KP) consists in selecting some items to load a knapsack with a total capacity W in order to maximize the total profit. The most common problem is called the $(0-1)$ knapsack because it selects at most one item and can be formulated as

$$KP \begin{cases} \max Z = \sum_{j=1}^{n} P_j X_j \\ \sum_{j=1}^{n} w_j X_j \leq W \\ X_j \in \{0,1\}\,; \quad j = 1, \cdots, n \end{cases}$$

Without loss of generality, the following conditions for the KP are usually assumed

(a) $P_j > 0, w_j > 0\,; \quad j = 1, \cdots, n$
(b) $w_j \leq \mathrm{W}\,; \quad j = 1, \cdots, n$
(c) $\sum_{j=1}^{n} w_j > \mathrm{W} > 0$

The zero-one knapsack problem is considered as the simplest linear programming problem and appears in many applications found in the industry and financial management [6, 10]. It is an NP combinatorial optimization problem and can obviously be solved by enumerating all possible subset of n items. This naïve approach with a complexity order of $\mathrm{O}(\mathbf{2}^{n})$ becomes inefficient when the number of items increases. Techniques of sampling [11] and parallel [8] computing have been developed to facilitate that approach for solving the KP.

Actually there are four main classes of algorithm solving the zero-one knapsack problem. Bellman [1] introduced in 1957 the first algorithm using dynamic programming which improved the complexity to $O(nW)$. Since then, numerous algorithms to solve the knapsack problem using dynamic programming have been developed [9]. However the capacity W can be an exponential function of n.

The second class of method uses the branch and bound algorithm which is firstly introduced in 1967 by Kolesar [7]. Between 1970 and 1979 many types of branch and bound algorithms were developed in order to solve large dimension KP [10]. The most well known approach of this period is due to Horowitz and Sahni [5]. In 1975, Sahni introduces the first polynomial time algorithm for the zero-one knapsack problem [13].

The algorithms of these first two classes are all exact methods while the last two are heuristics. The third class of algorithm provides near optimal solution. The most popular is the Greedy Algorithm introduced in 1957 by Dantzig [3]. The remaining class describes the evolutionary algorithms and particularly the genetic algorithms [4, 15] which behave very well when applied to some types of knapsack problem.

In this paper we introduce a fifth class which link the $(0-1)$ knapsack problem to the linear transportation problem. The resulting method is solved by an adaptation of the transportation algorithm and provides an optimal solution. The link between the knapsack problem and the transportation problem is the subject of the section 2. Then the adapted transportation algorithm is introduced in section 3. Finally, an example is presented for illustration.

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2. Knapsack Problem and Transportation

The zero-one knapsack problem can be formulated as a linear transportation problem. We associate to the knapsack the vector

$$X_{1,j} = X_j \;; \quad j = 1, \cdots, n$$

In order to have a transportation formulation, we proceed to a change of variables $Y_{ij} = w_j X_{ij}$ and consider a dummy knapsack 2 associated to variables

$$Y_{2,j} \;; \quad j = 1, \cdots, n$$

Their coefficients in the maximization objective are equal to zero, $K_{2,j} = 0$; $j = 1, \cdots, n$ which make these variables less attractive. Since the knapsack problem is not usually balanced, we also add a dummy surplus item $n+1$ with zero coefficients and its weight equal to an unknown buffer B which is the non-allocated knapsack capacity i.e. $w_{(n+1)} = B$. The total capacities of the knapsack and the dummy knapsack 2 are equal to

$$W_1 = W \quad \text{and} \quad W_2 = B + \sum_{j=1}^{n} w_j - W$$

All these transformations imply a balanced knapsack problem (BKP)

$$KP \begin{cases} \max Z = \sum_{j=1}^{n} \frac{P_j}{w_j} Y_{1,j} \\ \sum_{j=1}^{n+1} Y_{1,j} = W_1 \text{ and } \sum_{j=1}^{n+1} Y_{2,j} = W_2 \\ \sum_{i=1}^{2} Y_{i,j} = w_j \;;\; j = 1, \cdots, (n+1) \\ Y_{i,j} \in \{0, w_j\} \;;\; i = 1,2 \;;\; j = 1, \cdots, (n+1) \end{cases}$$

The items are ordered in the decreasing of the efficiency rate with the largest being

$$K_{1,1} = \max_j \left\{ \frac{P_j}{w_j} \right\}$$

Then we subtract all the coefficients from that largest value to obtain new coefficients

$$K_{1,j} = K_{1,1} - \frac{P_j}{w_j} \quad \text{and} \quad K_{2,j} = K_{1,1} \;; \quad j = 1, \cdots, n$$

This transforms the knapsack problem to the following minimization Transportation KP (TKP)

$$TKP \begin{cases} \min Z_L = \sum_{i=1}^{2} \sum_{j=1}^{n+1} K_{i,j} Y_{i,j} \\ \sum_{j=1}^{n+1} Y_{1,j} = W_1 \text{ and } \sum_{j=1}^{n+1} Y_{2,j} = W_2 \\ \sum_{i=1}^{2} Y_{i,j} = w_j \;;\; j = 1, \cdots, (n+1) \\ Y_{i,j} \in \{0, w_j\} \;;\; i = 1,2 \;;\; j = 1, \cdots, (n+1) \end{cases}$$

It can be solved by an adaptation of the transportation algorithm presented in the next section.

3. Adapted Transportation Algorithm

3.1. Initial solution

To obtain an initial solution, we use the Vogel Approximation Method (VAM). Since the items are ordered in the decreasing efficiency rate, it consists on assigning successively the item j if feasible. We note by V_j the remaining capacity after the assignment of the variable $Y_{1,j}$ and establish the procedure as follows.

Set $Y_{1,1} = w_1$ and $V_1 = W - w_1$
For $j = 2, \dots, n$
If $w_j \leq V_{(j-1)}$ then $Y_{1,j} = w_j$ and $V_j = V_{(j-1)} - w_j$
Else $Y_{1,j} = 0$, $Y_{2,j} = w_j$ and $V_j = V_{(j-1)}$
EndFor

Remark 1.The approach by the Vogel method is equivalent to the Greedy Algorithm for the knapsack problem with $O(n \ln(n))$ order of complexity.

3.2. Dual Variables and test of optimality

The dual variables u_i and v_j associated to the current solution is provided by the following system of equations

$$K_{i,j} - u_i - v_j = 0 \;; \quad \forall\, Y_{i,j} \in B_V$$

The set B_V is defined as the basic variables which induce $m + n - 1$ equations. Since there are $m + n$ unknown variables, by setting $u_1 = 0$ we can determine a solution of dual variables and reduced costs of all non-basic variables

$$\widehat{K}_{i,j} = K_{i,j} - u_i - v_j \;; \quad \forall\, Y_{i,j} \notin B_V$$

We can notice that the current solution might be improved if there exist at least one non-basic variables $Y_{i,j}$ such that

$$\widehat{K}_{i,j} = K_{i,j} - u_i - v_j < 0$$

In the case of the zero-one knapsack problem, the reduced costs of such non-basic variables verify

$$Y_{1,j} = 0,\; Y_{2,j} = w_j \quad \text{and} \quad \widehat{K}_{1,j} = K_{1,j} - K_{1,1} < 0$$

This indicates the possibility to decrease the objective cost by considering the move from $Y_{2,j}$ to $Y_{1,j}$ called the j-move which will decreases the cost by

$$R_j = \widehat{K}_{1,j}\ Y_{2,j} < 0\ .$$

Since the first constraint is limited by the knapsack capacity W the j-move implies the use of the remaining capacity which will not be sufficient because $V_j < w_j$. Therefore it will also need minimal necessary moves of variables $Y_{1,k}$ to $Y_{2,k}$ with $k < j$.

Proposition 1. If $Y_{1,k} = w_k > w_j$ then the move of $Y_{1,k}$ to $Y_{2,k}$ cannot be induced by any improving j-move of the weight w_j from the variable $Y_{1,j}$ to $Y_{2,j}$.

Proof. Since $Y_{1,k} = w_k > w_j$ and $K_{1,j} > K_{1,k}$ then

$$\left|\widehat{K}_{1,j}\right| = K_{1,1} - K_{1,j} < \widehat{K}_{2,k} = K_{1,1} - K_{1,k}$$

and

$$\widehat{K}_{2,k} \cdot X_{1,k} + \widehat{K}_{2,j} \cdot Y_{1,j} \geq 0$$

Therefore the move of $Y_{1,k}$ to $Y_{2,k}$ cannot be induced by any improving and feasible j-move.

Proposition 2. Let's consider $X_{1,k} = w_k$ and the move of the weight w_k from $Y_{1,k}$ to $Y_{2,k}$ such that

$$R_j + I_j + (\widehat{K}_{2,k} \cdot Y_{1,k}) \geq 0.$$

Then that move cannot be induced by any improving improving j-move.

Proof. If the move of $Y_{1,k}$ to $Y_{2,k}$ is induced by the j-move then the objective cost will increase by

$$R_j + I_j + \left(\widehat{K}_{2,k} \cdot Y_{1,k}\right) \geq 0$$

and will not improve it.

Let's define $U_j \geq V_j$ as the total capacity induced by all the admissible move of the basic variables to non-basic variables associated to an increase $\bar{I}_j$ of the objective function then we have the following remark.

Remark 2. Let's consider $Y_{1,k} = w_k$ such that

$$w_k + U_j < w_j \text{ and } R_j + \bar{I}_j + \left(\widehat{K}_{2,k} \cdot Y_{1,k}\right) < 0.$$

Then the move of the weight w_k from the basic variable $Y_{1,k}$ to $Y_{2,k}$ is admissible on the j-move.

Proposition 3. Let's consider $Y_{1,k} = w_k$ such that

$$w_k + U_j \geq w_j \text{ and } R_j + \bar{I}_j + \left(\widehat{K}_{2,k} \cdot Y_{1,k}\right) < 0.$$

Then the j-move is accepted. Moreover if the j-move is such that

$$w_k + U_j - w_j = 0$$

then the current solution yield by the j-move is optimal.

Proof. Since $w_k + U_j \geq w_j$ then the move of the basic variables associated to U_j provides a feasible j-move which yields an improved current solution. Therefore the j-move is accepted.

With the second condition, all variables $r > j$ are non-basic variables with negative reduced cost i.e.

$$Y_{1,r} = 0,\ X_{2,r} = w_r \text{ and } \widehat{K}_{1,r} = K_{1,r} - K_{1,1} < 0$$

Therefore, any r-move will involve the move of a total weight at least equal to w_r from basic variables $Y_{1,k}$; $k < j$ to $Y_{2,k}$. Since the minimal reduced cost associated to these moves are higher to the absolute value of the negative reduced cost of any non-basic variables $r > j$ then the objective function can no longer be improved.

These results motivate the improvement of the initial solution by using the approach organized and summarized in the following procedure.

Procedure 1. Improving the current solution

Let $Y_{1,\sigma(t)}$; $t = 1, \dots, T$ be the basic variables such that $\sigma(t) > j$

$$I_j = \sum_{t=1}^{T} \widehat{K}_{2,\sigma(t)} \cdot Y_{1,\sigma(t)} > 0$$

For $k = (j-1), \dots, 1$
If $w_k > w_j$ then continue
Else $(w_k \leq w_j)$
  If $w_k + V_j \geq w_j$ then
    if $R_j + I_j < 0$ then the $j-$ move is admissible.
    else the move of $X_{1,k}$ is discarded
  Else $(w_k + V_j < w_j)$
    If $w_k + U_j \geq w_j$ then
     if $R_j + \bar{I}_j < 0$ then the $j-$move is admissible.
     else the move of $Y_{1,k}$ is discarded
    Else $(w_k + U_j < w_j)$
     if $R_j + \bar{I}_j < 0$ then set $U_j = w_k + U_j$ and $\bar{I}_j = \bar{I}_j + \widehat{K}_{2,k} \cdot Y_{1,k}$
     else then the move of $Y_{1,k}$ is discarded
EndIf
EndFor

3.3. Adapted Transportation Algorithm

In this subsection, we present the adapted transportation algorithm

Step 0. **Listing the items**

List the items in the decreasing efficiency rate $\dfrac{P_j}{w_j}$

Update the coefficients to change the KP into minimization

$$K_{i,j} := K_{1,1} - K_{i,j};\quad i = 1,2\ ;\quad j = 1, \dots, n$$

Step 1. **Initial solution and reduced costs**

Set $Y_{1,1} = w_1, V_1 = W - w_1$ and s=0

For $j = 2, \dots, n$ do

If $w_j \le V_j$ then set $X_{1,j} = w_j$,

$V_{(j+1)} = V_j - w_j$ and $\widehat{K}_{2,j} = K - K_{1,j}$

Else set $Y_{2,j} = w_j, \widehat{K}_{1,j} = K_{1,j} - K$,

Set $V_{(j+1)} = V_j$, s:=s+1 and $TC(s) = j$

EndIf

Endfor

Set $Y_{1,(n+1)} = B = w_{(n+1)}$ and $Y_{2,(n+1)} = 0$

Step 2. Decreasing the objective function

For $k = 1, \dots, s$ do

Set $j = TC(k)$ and apply Procedure 1

If the j-move is accepted go to step 3

Else continue

EndFor

Go to step 4

Step 3. Update the current solution

If $w_k + U_j - w_j = 0$ then

the j-move is optimal and go to step 4

Else update the current solution and go to step 1

Step 4. Perform the optimal move

Find the optimal solution.

4. Example of Illustration

Let's consider the Knapsack Problem [14] defined by the following parameters

$$n = 7, \qquad (P_j) = (70, 20, 39, 35, 7, 5, 9)$$
$$(w_j) = (31, 10, 20, 18, 4, 3, 6) \quad \text{and} \quad W = 50$$

The KP can be represented by the following table

70	20	39	35	7	5	9	**50**
31	**10**	**20**	**18**	**4**	**3**	**6**	

To balance the problem with $W = 50 < \sum w_j = 92$, we need to add a dummy knapsack 2 and a dummy item 8 with its weight being equal to the unknown buffer B i.e. $w_8 = B$. The supply of the dummy knapsack 2 is equal to

$$W_2 = B + \sum_{j=1}^{7} w_j - W = B + 42$$

The coefficient associated to the dummy knapsack and dummy item are equal zero.

Now, by using the change of variable $Y_{i,j} = w_j X_{i,j}$ the associated table representation becomes

$\frac{70}{31}$	$\frac{20}{10}$	$\frac{39}{20}$	$\frac{35}{18}$	$\frac{7}{4}$	$\frac{5}{3}$	$\frac{3}{2}$	0	**50**
0	0	0	0	0	0	0	0	$\boldsymbol{B+42}$
31	**10**	**20**	**18**	**4**	**3**	**6**	$\boldsymbol{B}$	

By subtracting the coefficients from the highest efficiency $\frac{70}{31}$, the formulation becomes the minimization and yields to the following transportation problem

0	$\frac{8}{31}$	$\frac{191}{620}$	$\frac{175}{558}$	$\frac{63}{124}$	$\frac{55}{93}$	$\frac{47}{62}$	$\frac{70}{31}$	**50**
$\frac{70}{31}$	$\frac{70}{31}$	$\frac{70}{31}$	$\frac{70}{31}$	$\frac{70}{31}$	$\frac{70}{31}$	$\frac{70}{31}$	$\frac{70}{31}$	$\boldsymbol{B+42}$
31	**10**	**20**	**18**	**4**	**3**	**6**	$\boldsymbol{B}$	

To solve the transportation problem, we first find an initial solution by using the VAM.

Assignment 1. Since the largest penalty is $q_1 = \frac{70}{31}$, the variable to be assigned is

$$Y_{1,1} = 31 \quad \text{and} \quad V_1 = 50 - 31 = 19$$

Then column 1 is crossed out from the table.

Assignment 2. Since the efficiency rate is decreasing, the next variable to be assigned is

$$Y_{1,2} = 10 \quad \text{and} \quad V_2 = 19 - 10 = 9$$

The column 2 is cross out from the table. Since the remaining supply is lower than the weight w_3 and w_4 the associated variables Y_{13} and Y_{14} cannot be assigned and $V_3 = V_4 = 9$. Therefore, we set

$$Y_{2,3} = 20 \quad \text{and} \quad Y_{2,4} = 18$$

Then columns 3 and 4 are crossed out from the table.

Assignment 3. The next variable to be assigned is $Y_{1,5} = 4$ and $V_5 = 9 - 4 = 5$

Then the column 5 is crossed out from the table.

Assignment 4. The next variable to be assigned is $Y_{1,6} = 3$ and $V_6 = 5 - 3 = 2$

Then column 6 is crossed out from the table. Since $S_4 = 2 < w_7 = 6$ then the remaining variable Y_{17} cannot be assigned. We set $Y_{2,7} = 6$ and fill the remaining variable by setting

$$Y_{1,8} = 2\,;\ Y_{2,3} = 20\,;\ Y_{2,4} = 18\,;\ Y_{2,7} = 6\,;\ Y_{2,8} = 0$$

The initial solution is presented in the following table with the total profit

$$Z = 102 \quad \text{with} \quad B = 2$$

31	**10**	−1.95	−1.94	**4**	**3**	−1.5	**2**
2.26	2	**20**	**18**	1.75	1.67	**6**	**0**

The values of all basic variables are presented on bold face while the reduced costs occupy that space for the non-basic variables. Since $\widehat{K}_{3,1} = -1.95 < 0$, then the move of

$Y_{2,3} = 20$ to $Y_{1,3}$ will also implies the move of $Y_{1,1} = 31$ to $Y_{2,1}$. Since $Y_{2,3} = 20 < Y_{1,1} = 31$ the move of $Y_{2,3} = 20$ is discarded.

Since the next least reduced cost is $\widehat{K}_{4,1} = -1.94$, then the move of $w_4 = 18$ from $Y_{2,4}$ to $Y_{1,4}$ will reduce the objective by

$$R_4 = 18 \times (-1.94) = -35$$

This will forces at least the moves of $Y_{1,8} = 2$, $Y_{1,6} = 3$, $Y_{1,5} = 4$ and $Y_{1,2} = 10$ then implies an increase $I_4 = (10 \times 2) + (4 \times 1.75) + (3 \times 1.67) = 30$ Since the sum $R_4 + I_4 = -5 < 0$, then the move of $Y_{2,4} = 18$ can be performed to provide a new solution which, from proposition 1, is optimal with the total profit $Z^* = 105$.

31			**18**				**1**	50
	10	**20**		**4**	**3**	**6**	**1**	44

The example is used by Teghem [14] to illustrate the Branch and Bound method where 17 branching iterations and 6 backtracking are needed to reach the optimal solution. At first they prove that the optimal satisfies $X_1 = 1$ and $X_3 = 0$ which is equivalent to $Y_{1,1} = 31$ and $Y_{1,3} = 0$. Then they applied the Branch and Bound method to the remaining variables. So the Adapted Transportation Algorithm (ATA), using 2 tests on the initial solution in order to reach the optimal solution, seems very promising. However, more testing, complexity analysis and comparison are needed to get a true validation of the algorithm.

CONCLUSION

In this paper, we have presented the zero-one knapsack problem (KP) as a Linear Transportation Problem and provide a new approach to solve it. To find an initial solution the Vogel Method (VAM) is used and shown to be equivalent to a Greedy Algorithm for the knapsack problem. Then an Adapted Transportation Algorithm (ATA) is applied to find an optimal solution. The approach opens a new field of research which formulates the zero-one knapsack problem as a transportation problem. It can also be extended to some variants of the knapsack problem such as the subset-sum problem ($p_j = w_j$) [2], the bounded knapsack problem and the knapsack-like problems. It also can be generalized to solve the zero-one Multiple Knapsack Problem (MKP) and the multiple subset-sum problems.

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