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# SOLVING SINGULARLY PERTURBED PROBLEM USING CENTER OF GRAVITY INTERPOLATION COLLOCATION METHOD

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***Abstract –* In this paper, we use the center of gravity interpolation collocation method to the well-known problem which is called singularly perturbed problem. It means that we should find a more effective way to simulate the boundary layer effect. Through the results of the three examples in paper, we can make sure that the center of gravity interpolation collocation method is efficient and accurate method.**

**Key Word** -Singular perturbation problems, Boundary layer effect, Center of gravity interpolation collocation method, Differential matrix

## 1. INTRODUCTION

In the study of fluid mechanics and heat conduction problems in Engineering, we often need to solve the highest derivative with a small parameter singular perturbation boundary value problems. The solution of the singular perturbation boundary problems has the characteristics of rapid changes in a small range, which is also the boundary layer effect in singular perturbation problems.Therefore, a more effective numerical method is needed to simulate the effect of the boundary layer. And we found that we can take advantage of the great numerical stability of barycentric interpolation,and then encrypt the computing nodes, use the barycentric interpolation collocation method to simulate of the boundary layer effect of singular perturbation problems.

Singular perturbation of two order boundary value problem general form is:

$$\varepsilon y''(x)+p(x)y'(x)+q(x)y(x)=f(x)\,.\,x\in[a,b] \quad (1.1)$$

Initial conditions is $y(a)=y_a, y(b)=y_b$

In this function: $\varepsilon$ is a very small parameter $0<\varepsilon<<1$, p(x), q(x), f(x) are sufficiently smooth function which are definited in section [a,b].

## 2. The barycentric interpolation collocation method

### 2.1 Differential matrix barycentric interpolation

In order to use barycentric interpolation collocation method to solve(1.1), we need to know some point about differential matrix for barycentric interpolation.

For the nodes $a=x_1<x_2<\cdots<x_n=b$ in the section [a,b],the function u(x) exists $u_j=u(x_j)$, therefore, the center of gravity interpolation function of u(x) is

$$\mathrm{u(x)}=\sum_{\mathrm{j=1}}^{\mathrm{n}} L_j(x)u_j \quad (2.1)$$

where

$$\mathrm{L}_j(x)=\frac{\dfrac{\omega_j}{x-x_j}}{\displaystyle\sum_{k=1}^{n}\frac{\omega_k}{x-x_k}} \quad (2.2)$$

$$\omega_{\mathrm{j}}=1\Big/\prod_{j\neq k}(x_j-x_k),\ j=1,2,\cdots,n$$

where: (2.2) is the basis function of the center of gravity type interpolation; $\omega_j$ is thecenter of gravity interpolation, and it only depends on the distribution of interpolationnodes. The m order derivative of function u(x) at node $x_1,x_2,\cdots,x_n$ can be describedas

$$\mathrm{u}_i^{(m)}:=u^{(m)}(x_i)=\frac{d^m u(x_i)}{dx^m}=\sum_{j=1}^{n}L_j^{(m)}(x_i)u_j=\sum_{j=1}^{n}D_{ij}^{(m)}u_j, m=1,2,\cdots \quad (2.3)$$

and its matrix form is

$$\mathrm{u}^{(m)}=D^{(m)}u \quad (2.4)$$

where $u^{(m)}=[u_1^{(m)},u_2^{(m)},\cdots,u_n^{(m)}]^T$ is the m order derivative vector of function u(x) at nodes; $\mathrm{u}=[u_1,u_2,\cdots,u_n]^T$ is the function value at nodes; Matrix $D^{(m)}$ is called m order differential matrix, and its element is $D_{ij}^{(m)}=L_j^{(m)}(x_i)$.

In (2.2),both side multiply $x-x_i(i\neq j)$,then we can get

$$L_j(x)\sum_{k=1}^{n}\omega_k\frac{x-x_i}{x-x_k}=\omega_j\frac{x-x_i}{x-x_j} \quad (2.5)$$

In order to be easy , make it

$$s(x)=\sum_{k=1}^{n}\omega_k\frac{x-x_i}{x-x_k} \quad (2.6)$$

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Type (2.5) both sides seek derivatives of variables,we get

$$L_j'(x)s(x)+L_j(x)s'(x)=\omega_j\left(\frac{x-x_i}{x-x_j}\right)' \qquad (2.7)$$

$$L_j''(x)s(x)+2L_j'(x)s'(x)+L_j(x)s''(x)=\omega_j\left(\frac{x-x_i}{x-x_j}\right)'' \qquad (2.8)$$

Type(2.6) seeks derivative of the variable x,and substitute $x=x_i$ to it, we get

$$s(x_i)=\omega_i, s'(x_i)=\sum_{k\neq i}\frac{\omega_k}{x_i-x_k}, s''(x_i)=-2\sum_{k\neq i}\frac{\omega_k}{(x_i-x_k)^2} \qquad (2.9)$$

Let type(2.9) substitute to(2.7) and (2.8), we notice $L_j(x_j)=0, i\neq j$ , we get

$$L_j'(x_i)=\frac{\omega_j/\omega_i}{x_i-x_j}, j\neq i \qquad (2.10)$$

$$L_j''(x_i)=-2\frac{\omega_j/\omega_i}{x_i-x_j}\left(\sum_{k\neq i}\frac{\omega_k/\omega_i}{x_i-x_k}+\frac{1}{x_i-x_j}\right), j\neq i \qquad \mathbf{(2.11)}$$

for $i=j$, we find $\sum_{j=1}^{n}L_j(x)=1$, both side seek derivative,we will know $\sum_{j=1}^{n}L_j^{(m)}(x)=0$ , then we will get

$$L_i'(x_i)=-\sum_{j\neq i}L_j'(x_i), L_i''(x_i)=-\sum_{j\neq i}L_j''(x_i)$$

(2.12)

Then we can know first order differential matrix and two order differential matrix

$$D_{ij}^{(1)}=L_j'(x_i), D_{ij}^{(2)}=L_j''(x_i) \qquad (2.13)$$

Using mathematical induction, we can get the recurrence formula of the m order differential matrix

$$\begin{cases} D_{ij}^{(m)}=m\left(D_{ii}^{(m-1)}D_{ij}^{(1)}-\frac{D_{ij}^{(m-1)}}{x_i-x_j}\right), i\neq j \\ D_{ii}^{(m)}=-\sum_{j=1, j\neq i}^{n}D_{ij}^{(m)} \end{cases} \qquad (2.14)$$

**2.2 Using in singularly perturbed two order boundary value problem**

Using the mark of the differential matrix,the discrete form of(1.2) can be turned into a matrix as

$$(\varepsilon D^{(2)}+PD^{(1)}+Q)\ y=f \qquad (2.15)$$

Where

$$y=[y_1,\cdots,y_n]^T, P=diag[p(x_i)], Q=diag[q(x_i)]$$

$$f=[f(x_1),\cdots,f(x_n)]^T$$

Just use displacement method to exert boundary condition(1.2), then solve the algebraic equation(2.15), the numerical solution of the problem can be obtained.

**3. Numerical experiment**

Example 1 , Let's consider the following function.

$$\begin{cases} \varepsilon u''(x)+(1-\frac{1}{2}x)u'(x)-\frac{1}{2}u(x)=0, x\in[0,1] \\ u(0)=1 \\ u(1)=1 \end{cases}$$

The exact solution is $u_T(x)=\frac{1}{2-x}-\frac{1}{2}e^{-\frac{x-\frac{x^2}{4}}{\varepsilon}}$ . The result of reproducing center of gravity interpolation collocation method are shown in Figure 1，Table 1..

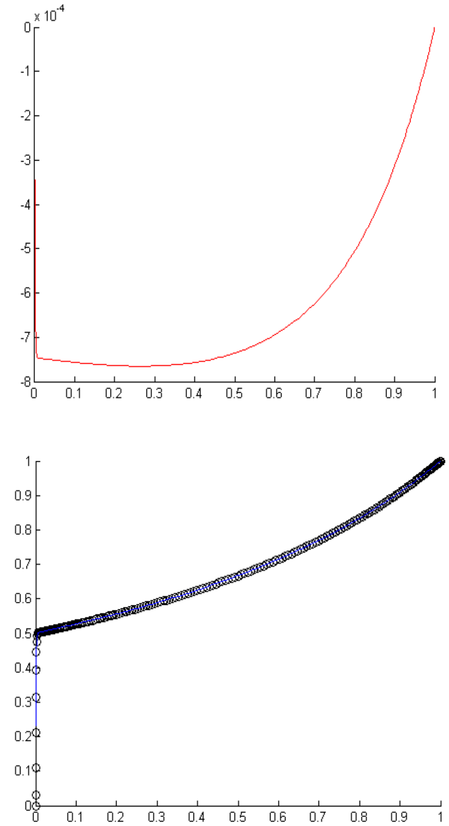

**Figure 1**: The left is relative error with center of gravity interpolation collocation method.The second one is numerical solution and exact solution when we choose $n=200, \varepsilon=10^{-4}$.

**Table 1. Comparison of the numerical results of Example**

| x | $u_T(x)$ | $u_{200}(x)$ | $\lvert u_T(x)-u_{200}(x)\rvert$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0.0001 | 0.0299 | 0.0300 | 0 |
| 0.0002 | 0.1094 | 0.1095 | 0.0002 |
| 0.0006 | 0.2131 | 0.2134 | 0.0003 |
| 0.0010 | 0.3138 | 0.3143 | 0.0005 |
| 0.0050 | 0.4978 | 0.4986 | 0.0007 |
| 0.0120 | 0.5030 | 0.5038 | 0.0007 |
| 0.0270 | 0.5068 | 0.5076 | 0.0007 |
| 0.0510 | 0.5131 | 0.5138 | 0.0008 |
| 0.5000 | 0.6667 | 0.6674 | 0.0007 |
| 1 | 1.0000 | 1.0000 | 0 |

Example 2, Considering the following one.

$$\begin{cases} \varepsilon u''(x)+u'(x)=1+2x, x\in[0,1] \\ u(0)=0 \\ u(1)=1 \end{cases}$$

The exact solution is $u_T(x)=x(x+1-2\varepsilon)+(2\varepsilon-1)\dfrac{1-e^{-\frac{x}{\varepsilon}}}{1-e^{-\frac{1}{\varepsilon}}}$. The result of reproducing center of gravity interpolation collocation method are shown in Figure 2.

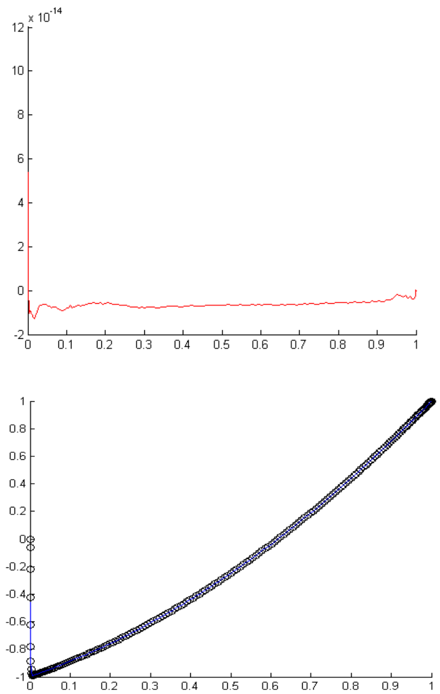


**Figure 2**: The first is relative error with center of gravity interpolation collocation method. The second one is numerical solution and exact solution when we choose $n=200, \varepsilon=10^{-4}$.

**Table 2. Comparison of the numerical results of Example 2**

| x | $u_T(x)$ | $u_{200}(x)$ | $\lvert u_T(x)-u_{200}(x)\rvert$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0.0001 | 0.0596 | 0.0596 | 0 |
| 0.0050 | 0.9862 | 0.9862 | 0 |
| 0.2800 | 0.6401 | 0.6401 | 0 |
| 0.3530 | 0.5211 | 0.5211 | 0 |
| 0.4140 | 0.4134 | 0.4134 | 0 |
| 0.5000 | 0.2490 | 0.2490 | 0 |
| 0.7200 | 0.2389 | 0.2389 | 0 |
| 0.8002 | 0.4409 | 0.4409 | 0 |
| 0.9999 | 0.9998 | 0.9998 | 0 |
| 1 | 1.0000 | 1.0000 | 0 |

Example 3, The last example.

$$\begin{cases} -\varepsilon u''(x)+u'(x)+(1+\varepsilon)u(x)=0, x\notin[0,1] \\ u(0)=1+e^{-\frac{1+\varepsilon}{\varepsilon}} \\ u(1)=1+e^{-1} \end{cases}$$

The exact solution is $u_T(x)=e^{\frac{(1+\varepsilon)(x-1)}{\varepsilon}}+e^{-x}$. The result of reproducing center of gravity interpolation collocation method are shown in Figure3.

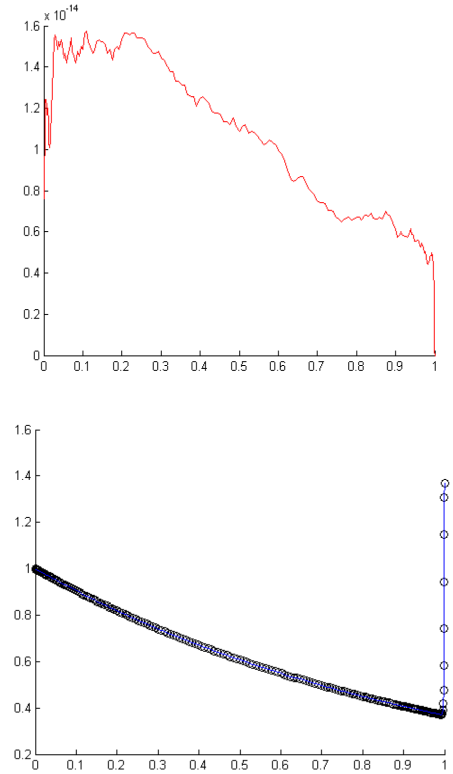


**Figure 3**: The first is relative error with center of gravity interpolation collocation method.The second one is numerical solution and exact solution when we choose $n=200, \varepsilon=10^{-4}$.

**Table 3. Comparison of the numerical results of Example 3**

| x | $u_T(x)$ | $u_{200}(x)$ | $\lvert u_T(x)-u_{200}(x)\rvert$ |
|---|---|---|---|
| 0 | 1.0000 | 1.0000 | 0 |
| 0.0001 | 0.9999 | 0.9999 | 0 |
| 0.1002 | 0.9047 | 0.9047 | 0 |
| 0.1520 | 0.8590 | 0.8590 | 0 |
| 0.2190 | 0.8034 | 0.8034 | 0 |
| 0.4140 | 0.6610 | 0.6610 | 0 |
| 0.5000 | 0.6065 | 0.6065 | 0 |
| 0.7200 | 0.4868 | 0.4868 | 0 |
| 0.8480 | 0.4283 | 0.4283 | 0 |
| 0.9999 | 1.3080 | 1.3080 | 0 |
| 1 | 1.3629 | 1.3629 | 0 |

## 4. CONCLUSIONS AND REMARKS

Singularly perturbed problem widely exists in engineering problems, because it is difficult to get a more effective way to imitate boundary layer effect. As people focus onthese problems, many numerical methods have been proposed.Through this paper, we found that center of gravity interpolation collocation method could simulate the boundary layer effect very well, and it is such an effective for singularly perturbed problem.

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## REFERENCES

[1] Berrut J P,Trerethen L N.Barycentric Language interpolation.SIAM Review, 46(2004),501-517.

[2] Berrut J P,A matrix for determining lower complexity barycenteic representations of rational interpolants .Numerical Algorithms,24(2004),17-29.

[3] Trefethen L N. Spectral Methods in Matlab.Philadelphia:SIAM(2004).

[4] Y.L.Wang,X.J.Cao,'A new method for solving singular fourth-order boundary value problems with mixed boundary conditions', Applied Mathematics and Computation **217**(2011),7385-7390.

[5] Y.L.Wang,M.J.Du,'Using reproducing kernel for solving a class of fractional partial dfference equation with non-classical conditions', Applied Mathematics and Computation.**219** (2013),5918-5925.

[6] Y.L.Wang,L.J.Su.'Using reproducing kernel for solving a class of singularly perturbed problems', Computers and Mathematics with Applications **61**(2011), 421-430.

[7] Y.L.Wang,C.L.Temuer,'New algorithm for second-order boundary value problems of integro-di□erential equation', Journal of Computational and Applied Mathematics **229** (2009), 1-6.

[8] Wang Y.L,Yu H,'Using an Effective Numerical Method for Solving a Class of Lane-Emden Equations', Abstract and Applied Analysis vol.2014, Article ID **735831**,8 pages,

[9] F.Z.Geng,Y.Z.Lin, ' Numerical solution of a system of fourth order boundary value problems using variational iteration method', Applied Mathematics and Computation **200** (2008), 231-241.

[10] Mehmet Cevik,M.Mustafa Bahs,'Solution of the delayed single degree of freedom system equation by exponential matrix method', Applied Mathematics and Computation **242**(2014), 444-453

[11] H.M.Yao,M.G.Cui, 'A new algorithm for a class of singular boundary value problems', Applied Mathematics and Computation **186** (2007), 1183-1191

[12] L.H.Yang,M.G.Cui,'New algorithm for a class of nonlinear integro-diifferential equations in the reproducing kernel space', Applied Mathematics and Computation **174** (2006), 942-960.

[13] X.Q.Lv,M.G.Cui, 'An efficient computational method for linear fifth-order two-point boundary value problems', Journal of Computational and Applied Mathematics **234** (2010), 1551-1558.

[14] M.G.Cui,Z.Chen,' The exact solution of nonlinear age-structured population model', Nonlinear Analysis: RealWorld Applications **8** (2007), 1096-1112.

[15] Wang Y, Zhao Y B,Wei G W.A note on the numerical solution of high-order differential equations.Journal of Computational and Applied Mathematics(2003),387-398.