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ON DIFFERENTIABILITY OF λ-RIESZ POTENTIALS AND THEIR APPLICATION.

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***Abstract-** In this work the λ-Riesz potentials are given and the problem of its totally differentiability is investigated. Numerical experiments support theoritical results.*

Keywords - Nonizotropic distance, differentiability

I. INTRODUCTION

Let $0<\alpha<n,\ x,y\in\mathbb{R}^n$, f belonging to $L_{loc}(\mathbb{R}^n)$ spaces.

$$R_\alpha f=\int_{\mathbb{R}^n} R_\alpha(x-y)f(y)dy$$

Define the integral operator called Riesz potential [1] where $R_\alpha(x-y)$ is the kernel of Riesz potential and defined

$$R_\alpha(x)=(x_1^2+\cdots+x_n^2)^{\frac{\alpha-n}{2}}.$$

For positive $\lambda_1,\ldots,\lambda_n, x\in\mathbb{R}^n$ and $|\lambda|=\lambda_1+\cdots+\lambda_n$

$$\|x\|_\lambda=\left(|x_1|^{\frac{1}{\lambda_1}}+\cdots+|x_n|^{\frac{1}{\lambda_n}}\right)^{\frac{|\lambda|}{n}}$$

is the non-isotropic distance between points x and 0 [2]. It is obvious that the Euclidean distance is a special case of this distance corresponding to $\lambda_j=\frac{1}{2},\ j=1,\ldots,n$.

For $0<\alpha<n$, let's define the Riesz potential depending on λ–distance as [3].

$$\Lambda_\alpha f(x)=\int_{\mathbb{R}^n}\|x-y\|_\lambda^{\alpha-n} f(y)dy$$

In this article, totally m times differentiability of λ-Riesz potentials is investigated. But , firstly, some basic properties of λ- Riesz potentials are given [2], [4], [5].

Lemma 1.1. For $k\in\mathbb{N}$, $j=1,\ldots,n$

$$\frac{\partial^k}{\partial x_i^k}\|x\|_\lambda^{\alpha-n}=\sum_{s=1}^{n}\left(\operatorname{sgn} x_j\right)^s|x_j|^{\frac{s}{\lambda_j}-k}|x|_\lambda^{\frac{|\lambda|}{n}(\alpha-n)-s}$$

where

$$\left(\operatorname{sgn} x_j\right)^s=(\operatorname{sgn} x_1)^{s_1}\cdots(\operatorname{sgn} x_n)^{s_n},\quad \|x\|_\lambda^{\frac{n}{|\lambda|}}=|x|_\lambda.$$

II. RESULTS

For any positive α and $j=1,\ldots,n$

$i)\quad |x_j|^\alpha\le\|x\|_\lambda^{\alpha\lambda_j\frac{n}{|\lambda|}}$

$ii)\quad |x|^\xi\le\|x\|_\lambda^{\xi|\lambda_j|\frac{n}{|\lambda|}}$

where $|\xi\lambda|=\xi_1\lambda_1+\cdots+\xi_n\lambda_n$, $|x|^\xi=|x_1|^{\xi_1}\cdots|x_n|^{\xi_n}$.

Lemma 2.1. For a positive natural m denote

$$K_m(x,y)=E_\alpha(x-y)-\sum_{|\xi|\le m}(\xi!)^{-1}x^\xi\left[\left(\frac{\partial}{\partial x}\right)^\xi E_\alpha\right](-y)$$

where $y\in\mathbb{R}^n-B_\lambda\left(0,2\|x\|_\lambda\right)$. Then there exist a positive constant C such that

$$|K_m(x,y)|\le\sum_{|\xi|=m+1}C\|x\|_\lambda^{\frac{n}{|\lambda|}|\xi\lambda|}\|y\|_\lambda^{(\alpha-n)-\frac{n}{|\lambda|}|\xi\lambda|}$$

where $E_\alpha(x-y)=\|x-y\|_\lambda^{\alpha-n}$ is the kernel of λ- Riesz potentials and $B_\lambda(x,r)$ is the λ-ball centered at x with radius r. That is

$$B_\lambda(x,r)=\left\{y\in\mathbb{R}^n:\|x-y\|_\lambda\le r\right\}.$$

Lemma 2.2. The integral

$$\int_{B_\lambda(x_0,1)}E_\alpha(x-y)dy$$

is infinitely differentiable in $B_\lambda(x_0,1)$.

Lemma 2.3. Let f be a nonnegative and locally integrable function in $\mathbb{R}^n$. In order $\Lambda_\alpha f(x)\ne\infty$ it is necessary and sufficient that

$$\int_{\mathbb{R}^n-B_\lambda(x,1)}\|x-y\|_\lambda^{\alpha-n} f(y)dy<\infty \qquad (2.1)$$

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for some $x \in \mathbb{R}^n$, moreover (2.1) is equalent to

$$\int_{\mathbb{R}^n} \left(1+\|y\|_\lambda\right)^{\alpha-n} f(y)\,dy < \infty$$

Lemma 2.4. Let f be a nonnegative measurable function in $\mathbb{R}^n$, $\alpha p = n$ and $pp' = p + p'$. Then there exist a positive constant M such that for any $a > 0$

$$\int_{\{y: f(y)\geq a\}} \Lambda_\alpha f(y)\,dy \leq M \left(\int_a^\infty \omega^{-\frac{1}{p-1}}(t)\,t^{-1}dt \right)^{\frac{1}{p}}$$

$$\cdot \left(\int_{\{y: f(y)\geq a\}} f^p(y)\,\omega^{\frac{1}{p}}\left(f(y)\right)dy \right)^{\frac{1}{p}}$$

Definition 2.5. The function u is totally m times differentiable at point x_0 if there exist a polinomial $P(x)$ of degree at most m such that

$$\lim_{x\to x_0} \|x-x_0\|_\lambda^{-m\Lambda\frac{n}{|\lambda|}} \left\{u(x)-P(x)\right\} = 0$$

where

$$\Lambda = \begin{cases} 1 & , \ \min\limits_{1\leq j\leq n} \lambda_j > 1 \\ \lambda_{\min} & , \ \min\limits_{1\leq j\leq n} \lambda_j < 1 \end{cases}$$

Theorem 2.6. Let $f \in L_{loc}\left(\mathbb{R}^n\right)$, $\alpha p = n$ and the following conditions hold:

$$\int_{\mathbb{R}^n} \left(1+\|y\|_\lambda\right)^{\alpha-n} f(y)\,dy < \infty$$

$$\int_{\mathbb{R}^n} f^p(y)\,\omega\left(f(y)\right)dy < \infty$$

where ω is function from Lemma 2.4. Let also

$$E_1 = \left\{ x \in \mathbb{R}^n : \lim_{r\to 0} r^{2\left[(\alpha-n)\frac{|\lambda|}{n} - \lambda_{\max} m\right]} \cdot \int_{B_\lambda(x,r)} \left|f(y)-f(x_0)\right| dy > 0 \right\}$$

$$E_2 = \left\{ x \in \mathbb{R}^n : \limsup_{r\to 0} r^{2\left[(\alpha-n)\frac{|\lambda|}{n} - \lambda_{\max}(m+1)\right]} \cdot \int_{B_\lambda(x,r)} \left| f^p(y)\,\omega\left(f(y)\right) - f^p(x)\,\omega\left(f(x)\right)\right| dy > 0 \right\}$$

$$E_3 = \left\{ x \in \mathbb{R}^n : \lim_{r\to 0} \int_{\mathbb{R}^n - B_\lambda(x,r)} \left(\frac{\partial}{\partial x}\right)^\xi E_\alpha(x-y)\left[f(y)-f(x_0)\right]dy = \infty \right\}$$

Then the λ -Riesz potentials are totally m times differentiable at any point of set $\mathbb{R}^n - E$ where $m \leq \alpha$ and $E = E_1 \cup E_2 \cup E_3$.

Proof. Using the function $K_m(x,y)$ from lemma 2.1. for

$$E_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)\,dy}{\|x-y\|_\lambda^{n-\alpha}},$$

$$\left\{E_\alpha f(x) - P(x)\right\} = \int_{\mathbb{R}^n - B_\lambda(x_0,1)} K_m(x,y) f(y)\,dy$$

$$+ \int_{B_\lambda(x_0,1) - B_\lambda\left(x_0, 2\|x-x_0\|_\lambda\right)} K_m(x,y)\left[f(y)-f(x)\right]dy$$

$$-\sum_{|\xi|=m} (\xi!)^{-1}(x-x_0)^\xi \lim_{r\to 0} \int_{B_\lambda(x_0,2\|x-x_0\|)-B_\lambda(x_0,r)} \left[\left(\frac{\partial}{\partial x}\right)^\xi E_\alpha\right](x_0-y)\,dy + f(x_0)\int_{B_\lambda(x_0,1)} \Big[K_{m-1}(x,y) -$$

$$\sum_{|\xi|=m} (\xi!)^{-1}(x-x_0)^\xi \left[\left(\frac{\partial}{\partial x}\right)^\xi E_\alpha\right](x_0-y)\Big]dy$$

$$+ \int_{B_\lambda\left(x_0,2\|x-x_0\|_\lambda\right)} E_\alpha(x-y)\left[f(y)-f(x_0)\right]dy$$

$$= I_1 + I_2 + I_3 + I_4 + I_5$$

where

$$P(x) = \sum_{|\xi|\leq m} (\xi!)^{-1}(x-x_0)^\xi \lim_{r\to 0} \int_{\mathbb{R}^n - B_\lambda(x_0,r)} \left[\left(\frac{\partial}{\partial x}\right)^\xi E_\alpha\right](x_0-y) f(y)\,dy + \sum_{|\xi|\leq m} (\xi!)^{-1}(x-x_0)^\xi$$

$$\lim_{r\to 0} \int_{B_\lambda(x_0,r)} \left[\left(\frac{\partial}{\partial x}\right)^\xi E_\alpha\right](x_0-y) f(x_0)\,dy.$$

Consider I_1 . Since in this integral domain $\|x-y_0\|_\lambda > 1$, by lemma 2.1. We have

$$I_1 \le \lim_{x\to x_0} \|x-x_0\|_\lambda^{m\frac{n}{|\lambda|}(\lambda_{\min}-\Lambda)+\lambda_{\min}\frac{n}{|\lambda|}}$$

$$\int_{\mathbb{R}^n - B_\lambda(x_0,1)} \|y-x_0\|_\lambda^{\alpha-n} f(y)\,dy$$

$m\frac{n}{|\lambda|}(\lambda_{\min}-\Lambda)+\lambda_{\min}\frac{n}{|\lambda|} > 0$ and from (2.1) it follows that this integral is finite an so

$\lim_{x\to x_0} \|x-x_0\|^{-\Lambda m\frac{n}{|\lambda|}} I_1 = 0$.

Now by lemma 2.1 we obtain ($\Lambda = \lambda_{\min}$)

$$\|x-x_0\|_\lambda^{-\lambda_{\min} m\frac{n}{|\lambda|}} I_2 \le \|x-x_0\|_\lambda^{\lambda_{\min} m\frac{n}{|\lambda|}} \int_{B_\lambda(x_0,1)-B_\lambda\left(x_0,2\|x-x_0\|_\lambda\right)}$$

$$\sum_{|\xi|=m+1} \|y-x_0\|_\lambda^{\frac{n}{|\lambda|}(\alpha-n)-|\xi\lambda|} \left[f(y) - f(x_0) \right] dy$$

$$\le \|x-x_0\|_\lambda^{\lambda_{\min} m\frac{n}{|\lambda|}} \rho^{2\left[\frac{n}{|\lambda|}(\alpha-n)-m\lambda_{\max}-\lambda_{\max}\right]} F(\rho)\Big|_{2\|x-x_0\|_\lambda}^{1}$$

$$-\int_{2\|x-x_0\|_\lambda}^{1} F(\rho)\, d\rho^{2\left[\frac{n}{|\lambda|}(\alpha-n)-m\lambda_{\max}\right]}$$

where

$$F(\rho) = \int_0^\rho \int_{s^{n-1}} \left[f(x_0 - t\theta) - f(x_0) \right] \left| I_\lambda(t,\theta) \right| dt d\theta,$$

$$I_\lambda(t\theta) = \rho^{2\|\lambda\|-1} \Omega_\lambda(\theta),$$

$$\Omega_\lambda(\theta) = 2^n \lambda_1 \cdots \lambda_n (\cos\theta_1)^{2\lambda_1 - 1} \ldots (\cos\theta_{n-1})^{2\lambda_{n-1}-1}$$

$$\cdot (\sin\theta_1)^{2(\lambda_2\cdots\lambda_n)-1} (\sin\theta_2)^{2(\lambda_3\cdots\lambda_n)-1} \cdots (\sin\theta_{n-1})^{2\lambda_n - 1}$$

[2]. Since $f \in L_{loc}(\mathbb{R}^n)$ and from the definition of E_2 we see that

$\lim_{x\to x_0} \|x-x_0\|_\lambda^{-\Lambda_m} I_2 = 0$.

Using the lemma 2.1 for derivatives of kernel $E_\alpha(x-y)$ we obtain

$$\lim_{x\to x_0} \|x-x_0\|_\lambda^{-\Lambda m\frac{|\lambda|}{n}} I_3$$

$$\le f(x_0) \sum_{|\xi|=m} (\xi!)^{-1} \|x-x_0\|_\lambda^{(\lambda_{\min}|\xi|+1)\frac{n}{|\lambda|}}$$

$$\cdot \int_{B_\lambda(x_0,1)} \left(\frac{\partial}{\partial x}\right)^{m+1} E_\alpha(s-y)\,dy$$

and from the definition E_2 we see that for any point

$x_0 \in \mathbb{R}^n - E_2$

$\lim_{x\to x_0} \|x-x_0\|_\lambda^{-\Lambda m\frac{|\lambda|}{n}} I_3 = 0$.

Consider the term I_4 . Applying the mean-value theorem we have

$$\|x-x_0\|_\lambda^{-\Lambda m\frac{|\lambda|}{n}} I_4 \le \|x-x_0\|_\lambda^{\left[(\alpha-n)\frac{|\lambda|}{n}-\lambda_{\min} m\right]\frac{|\lambda|}{n}}$$

$$\cdot \int_{B_\lambda\left(x_0,2\|x-x_0\|_\lambda\right)-B_\lambda(x_0,1)} \left| f(y) - f(x_0) \right| dy$$

and lemma 2.2 the integral on the right side finite and

$\lim_{x\to x_0} \|x-x_0\|_\lambda^{-\Lambda m\frac{|\lambda|}{n}} I_4 = 0$.

Finally consider the term I_5 . For

$y \in B_\lambda\left(x_0, 2\|x-x_0\|_\lambda\right)$ and $x \to x_0$ we have

$$\|x-x_0\|_\lambda^{-\Lambda m\frac{|\lambda|}{n}} I_5 = \|x-x_0\|_\lambda^{-\Lambda m\frac{|\lambda|}{n}}$$

$$\cdot \int_{B_\lambda\left(x_0,2\|x-x_0\|_\lambda\right)} \|x-y\|_\lambda^{(\alpha-n)\frac{|\lambda|}{n}} \left[f(y) - f(x_0) \right] dy$$

$$= \|x-x_0\|_\lambda^{-\Lambda m\frac{|\lambda|}{n}} \int_{\left\{y\in B_\lambda\left(x_0,2\|x-x_0\|_\lambda\right):\psi(y)<a\right\}} \psi(y) \|x-y\|_\lambda^{(\alpha-n)\frac{|\lambda|}{n}}\, dy$$

where a is any positive number and

$\psi(y) = f(y) - f(x_0)$.

The first integral may be easily calculated and since $m \le \alpha$ the first term tends to zero as $x \to x_0$. Applying lemma 2.4 to second term on righthand side we can see that by definition of set E_3 . The second term also tends to zero as $x \to x_0$ for any $x_0 \in \mathbb{R}^n - E_3$. The prof is completed.

III EXAMPLES

In this section, theoritical results is supported examples. The following potential

$$I = \iint_D \frac{\sin(xy)}{\left(|x-y|^{\frac{1}{\lambda_1}} + |x-y|^{\frac{1}{\lambda_2}} \right)^{\frac{(\lambda_1+\lambda_2)(n-\alpha)}{n}}} dxdy$$

was considered in domain

$D = \{(x,y): 0 \le x \le 1,\ 2 \le y \le 3\}$.

This integral was evaluatedfor various λ_1, λ_2 and α values in two dimensional space by Simpson Formula. The choices $N = 20$ and $M = 20$ are used in Simpson Formula. The results indicates as following: for $\lambda_1 = 2,\ \lambda_2 = 3$ and $\alpha = 1.5$ integral $I \approx 0.21558873$, $\lambda_1 = \frac{1}{2},\ \lambda_2 = \frac{1}{4}$ and $\alpha = 1.5$ integral $I \approx 0.42459130$. Furthermore, at figure 1 and 2, one is drawn graphic of integrand for these values by mathematica:

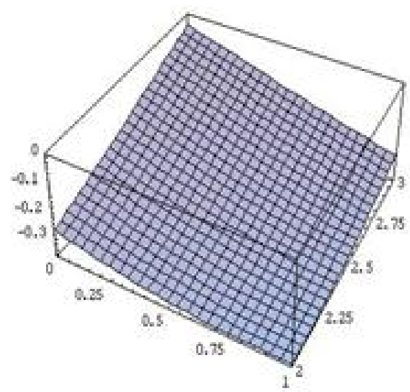

Figure 1. The graphic of integrand for $\lambda_1 = 2, \lambda_2 = 3$ and $\alpha = 1.5$

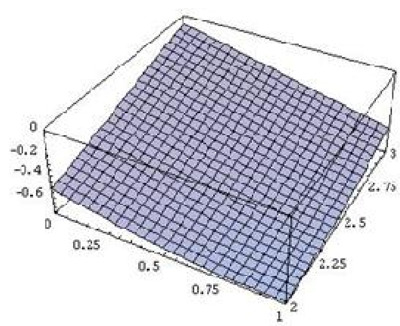

*Figure 2.*The graphic of integrand for $\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{4}$ and $\alpha = 1.5$

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