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EL-GENDI NODAL GALERKIN METHOD FOR SOLVING LINEAR AND NONLINEAR PARTIAL FRACTIONAL SPACE EQUATIONS

[1,*] M. El-Kady, [2]Salah M. El-Sayed, [3,*] Heba. E. Salem
[1]Department of Mathematics, Faculty of Science, Helwan University, Cairo, Egypt
[2]Department of Scientific Computing, Faculty of Computers and Informatics, Benha University, Benha 13518, Egypt
[3]Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt

***Abstract- In this paper, an efficient numerical technique is presented to solve the partial fractional space equations with variable coefficients on a finite domain. This technique based on nodal Galerkin method. The fractional derivatives are described in the Caputo sense. Also, a fully discrete scheme is given for a type of nonlinear space-fractional anomalous advection-diffusion equation. In this paper, the problems can be reduced to a set of linear algebraic equations by using the Chebyshev nodal Galerkin method. The existence and uniqueness of the solution for the linear semi discrete weak form solutions are proved. And the stability analysis for the linear semi and fully discrete schemes are discussed. Numerical solutions obtained by this method are in excellent agreement and efficient to use with those obtained by previous work in the literature*.**

Keywords - Shifted Chebyshev polynomial; Nodal Galerkin method; Fractional diffusion equation; Caputo derivative

1. INTRODUCTION

In recent years, a lot of attention has been devoted to the study of fractional differential equations. Fractional derivatives arise in many physical and engineering problems such as electric transmission, ultrasonic wave propagation in human cancellous bone, modeling of speech signals, modeling the cardiac tissue electrode interface, viscoelasticity, wave propagation in viscoelastic horns and fluid mechanics [13] and [3].

In this paper, we present a direct computational technique for the one-dimensional space fractional diffusion equation of the form:

$$u_t = \lambda(x)\,{}_aD_x^{\gamma+1}u + f(x,t),$$
$$a \le x \le b, 0 < \gamma \le 1,\ 0 \le t \le T, \qquad (1)$$

with initial and homogenous boundary conditions as follows:

$$u(x,0) = \varphi(x), \qquad a \le x \le b,$$
$$u(a,t) = u(b,t) = 0, \quad 0 \le t \le T, \qquad (2)$$

where the anomalous item ${}_aD_x^{\gamma}u$ is the γ th order fractional derivative of u with respect to the space variable x in the Caputo sense which will be introduced later on. We always consider: $0 < \eta_1 < \lambda(x) < \eta_2$, where η_1, η_2 are constants.

The fractional order diffusion equations are generalizations of classical diffusion equations. These equations play important roles in modeling anomalous diffusion and sub-diffusion systems, description of a fractional random walk, unification of diffusion and wave propagation phenomena, see, e.g., [20], and the references therein. Many numerical investigations were carried out by many authors to solve this problem. In [2] the backward Euler finite difference scheme is applied in order to obtain numerical solutions for the equation. Existence and stability of the approximate solutions are carried out by using the right shifted Grünwald formula for the fractional derivative term in the spatial direction. In [21] approximation techniques based on the shifted Legendre-tau idea are presented to solve a class of initial-boundary value problems for the fractional diffusion equations. The technique is derived by expanding the required approximate solution as the elements of shifted Legendre polynomials. In [10] Legendre pseudo-spectral method with the finite difference method is used to obtain the numerical solution of the fractional diffusion equation. Also, we mainly study one kind of typical nonlinear space-fractional partial differential equations which is called fractional anomalous diffusion and has the following form:

$$w_t = D_x(a(w)\,{}_aD_x^{\gamma}w) + \rho(x)D_x w + f(x,t,w),$$
$$a \le x \le b, \gamma \in (0,1),\ 0 \le t \le T, \qquad (3)$$

with initial and boundary conditions as follows:

$$w(x,0) = \psi(x), \qquad a \le x \le b,$$
$$w(a,t) = w(b,t) = 0, \quad 0 \le t \le T, \qquad (4)$$

where ${}_aD_x^{\gamma}$ is the α th order fractional derivative with respect to the space variable x in the Caputo sense. Now the fractional anomalous diffusion becomes a hot topic because of its widely applications in the evolution of various dynamical systems under the influence of stochastic forces.

*Corresponding authors: 5015, St. #17, Mokattam, Cairo, Egypt.
E-mail: mam_el_kady@yahoo.com
*E-mail: drheba783@gmail.com

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For example, it is a well-suited tool for the description of anomalous transport processes in both absence and presence of external velocities or force fields. Moreover, the fractional anomalous diffusion have numerous applications in statistical physics, biophysics, chemistry, hydrogeology, and biology, see for more details [8],[16] and [17]. There are some authors studying the special anomalous diffusion equation in theoretical analysis and numerical simulations, see [6], [14] and [23].

In this paper, we used El-gendi nodal Galerkin method which is easier technique than the usual Galerkin method. In Galerkin method, each basis polynomial chosen must satisfy the boundary conditions individually which causes the Galerkin formulation to become complicated, particularly when the boundary conditions are time-dependent [1]. Furthermore, the presence of nonlinear term complicates the computation of the stiffness matrix [9]. However, the Galerkin method is based on a variation formulation which preserves essential properties of the continuous problem such as coercively, continuity and symmetry of the bilinear form and it usually leads to optimal error estimates [22].

On the other hand, the main advantage of the nodal Galerkin method is its simplicity and flexibility in implementation. In addition, this method deals with nonlinear terms more easily than Galerkin methods. Moreover, the problems with variable coefficients and general boundary conditions are treated as the same way as problems with constant coefficients and simple boundary conditions. In fact, In El-gendi Chebyshev nodal Galerkin method, we start from a weak form of the equations, but we replace hard to evaluate integrals by El-gendi quadrature. The formula of El-gendi quadrature is satisfying a symmetric property. Hence, we can reduce the number of operations to 50% which implies to decrease the rounding error. Also, El-gendi quadrature is an alternating series which converges as $N \to \infty$ (N is the number of grid points).

The remainder of this paper is organized as follows: In section 2, we present the procedure of solution for the partial fractional space equation in a linear and nonlinear case. In section 3, we present the error analysis. In section 4, we give numerical experiments to clarify the method.

2. Fractional Derivative Space

In this section we will give the fractional derivative space. Firstly, we will give the following definitions:

Definition 1. The fractional derivative in the Riemann-Liouville version of function $f(x)$ is defined as follows [19].

$$ {}_aJ_x^{\gamma} f(x) = \frac{1}{\Gamma(n-\gamma)} \frac{d^m}{dx^m} \int_a^x \frac{f(s)}{(x-s)^{\gamma+1-m}} ds, $$

where $m-1 < \gamma < m, m \in N$.

An alternative definition, known as the Caputo fractional derivative is:

$$ {}_aD_x^{\gamma} f(x) = \frac{1}{\Gamma(m-\gamma)} \int_a^x \frac{f^{(m)}(s)}{(x-s)^{\gamma+1-m}} ds. \tag{5} $$

The two definitions are not in general equivalent but they are related by the following relation:

$$ {}_aD_x^{\gamma} f(x) = {}_aJ_x^{\gamma} f(x) - \sum_{k=0}^{m-1} \frac{x^{k-\gamma} f^{(k)}(0)}{\Gamma(k+1-\gamma)}. $$

Generally, when we consider the fractional differential equations the Caputo definition is often preferred since it is easy for imposing initial and boundary conditions on classic derivatives. But for the Riemann-Liouville definition, these conditions must be imposed on fractional derivatives and this is often not available. So that, we will use the Caputo definition in this paper.

Definition 2. [19] For $\gamma > 0$, the fractional derivative space $I^{\gamma}(a,b)$ is defined as follows:

$$ I^{\gamma}(a,b) = \{ f \in L^2(a,b) : {}_aD_x^{\gamma} f \in L^2(a,b), \; m-1 \le \gamma \le m \}, $$

endowed with the semi-norm:

$$ |f|_{I^{\gamma}(a,b)} = \| {}_aD_x^{\gamma} f \|_{L^2(a,b)}, $$

and the norm

$$ \|f\|_{I^{\gamma}(a,b)} = \left(|f|^2_{I^{\gamma}(a,b)} + \|f\|^2_{L^2(a,b)} \right)^{1/2}, $$

and let $I^{\gamma}(a,b)$ denotes the closure of $C_0^{\infty}(a,b)$ with respect to the above norm and seminorm.

Definition 3. [15] The fractional space $E^{\gamma}(a,b)$ defined below

$$ E^{\gamma}(a,b) = \{ f \in L^2(a,b) : {}_aD_x^{\gamma} f \in L^2(a,b), \; {}_xD_b^{\gamma} f \in L^2(a,b), m-1 \le \gamma < m \}, $$

endowed with the seminorm

$$ |f|_{E^{\gamma}(a,b)} = \left| \left({}_aD_x^{\gamma} f, \; {}_xD_b^{\gamma} f \right) \right|^{1/2}, $$

and the norm

$$ \|f\|_{E^{\gamma}(a,b)} = \left(|f|^2_{E^{\gamma}(a,b)} + \|f\|^2_{L^2(a,b)} \right)^{1/2}, $$

and let $E^{\gamma}(a,b)$ denotes the closure of $C_0^{\infty}(a,b)$ with respect to the above norm and seminorm.

Definition 4. [7] For $\gamma > 0$, define the seminorm

$$ |f|_{H^{\gamma}(a,b)} = \left\| |\omega|^{\gamma} F(f) \right\|_{L^2(a,b)}, $$

and the norm

$$ \|f\|_{H^{\gamma}(a,b)} = \left(|f|^2_{H^{\gamma}(a,b)} + \|f\|^2_{L^2(a,b)} \right)^{1/2}, $$

where $F(f)$ is the Fourier transform of the function f and which can define another fractional derivative space $H^{\gamma}(a,b)$. Let $H_0^{\gamma}(a,b)$ be the closure of $C_0^{\infty}(a,b)$ with respect to the above norm and seminorm.

Theorem 1. [7] The spaces $I^{\gamma}(a,b)$, $E^{\gamma}(a,b)$ and $H_0^{\gamma}(a,b)$ are equal in the sense that their semi norms as well as norms are equivalent.

Lemma 1. [6 (Fractional Poincaré–Friedrichs)]
For $f \in H_0^{\gamma}(a,b)$, we have

$$\|f\|_{L^2(a,b)} \le C|f|_{H_0^{\gamma}(a,b)},$$

and for $0<\alpha<\gamma$, $\alpha \ne m-1/2$, $m \in Z^+$,

$$|f|_{H_0^{\alpha}(a,b)} \le C|f|_{H_0^{\gamma}(a,b)}.$$

Lemma 2. [7] For $f \in I_0^{\gamma}(a,b)$, $0<\alpha<\gamma$, then

$${}_aD_x^{\gamma} f(x) = {}_aD_x^{\gamma-\alpha}\, {}_aD_x^{\alpha} f(x).$$

3. NUMERICAL TREATEMENTS FOR THE PARTIALFRACTIONAL SPACE

In this section, we present the numerical solution for time–space fractional linear and nonlinear equations, respectively, where the space fractional derivative is the Caputo derivative.

3.1 EL-GENDI NODAL GALERKIN METHOD FOR LINEAR CASE

This method starts with the weak form and the trail space coincides with the test function space. The weak form of problem (1) and (2) in case $a=0, b=L$ is given as follows:

Find $u \in H_0^{(1+\gamma)/2}(0,L)$ such that

$$(u_t, v) = -\left(({}_0D_L^{\gamma}u), D_x(\lambda(x)v)\right) + \left(f(x,t), v\right),$$
$$\forall v \in H_0^{(\gamma+1)/2}(0,L),\ t>0, \tag{7}$$

where the inner product (u,v) is defined as

$$(u,v) = \int_0^L u(x)v(x)\,dx.$$

Next, we will prove the existence and uniqueness of the weak form (7). So, we give the properties of the fractional diffusion operator which is given in [14] as follows:

1- $\left(({}_0D_L^{\gamma+1}u), u\right) = \left(({}_0D_L^{\gamma}u), D_x u\right) \ge \sigma_1 \|u\|^2_{H^{(\gamma+1)/2}}$ coercivity on $H_0^{(\gamma+1)/2}(0,L)$,

2- $$\left(({}_0D_L^{\gamma+1}u), v\right) = \left(({}_0D_L^{\gamma}u), D_x v\right) \le \sigma_2 \|u\|^2_{H^{(\gamma+1)/2}} \|v\|^2_{H^{(\gamma+1)/2}}$$

continuity on $H_0^{(\gamma+1)/2}(0,L) \times H_0^{(\gamma+1)/2}(0,L)$, where σ_1, σ_2 are constants.

Applying the implicit Euler approximation to approximate the time derivative, we define

$t_\ell = \ell\,\Delta t,\ 0 \le t_\ell \le T = 1,2,\ldots$ and Δt is the time step.

Then equation (7) is approximated as follows: Find $u^{\ell+1} \in H_0^{(1+\gamma)/2}(0,L)$ such that

$$\left(u^{\ell+1}, v\right) + \Delta t\left(({}_0D_L^{\gamma}u^{\ell+1}), D_x(\lambda(x)v)\right) = \left(u^{\ell}, v\right) + \Delta t\left(f^{\ell+1}, v\right)\ \ \forall v \in H_0^{(\gamma+1)/2}(0,L),\ t>0, \tag{8}$$

where $f^{\ell+1} = f(x, t_{\ell+1})$. Let

$$B(u^{\ell+1}, v) = \left(u^{\ell+1}, v\right) + \Delta t\left(({}_0D_L^{\gamma}u^{\ell+1}), D_x(\lambda(x)v)\right),$$

and

$$F(v) = \left(u^{\ell} + \Delta t\, f^{\ell+1}, v\right) = (g, v),$$

then the semi-discrete problem (8) can be written in a simple form like that:

$$B(u^{\ell+1}, v) = F(v),\ \ \forall v \in H_0^{(\gamma+1)/2}(0,L),\ t>0, \tag{9}$$

Theorem 2 (Existence and Uniqueness).

For $0<\eta_1<\lambda(x)<\eta_2$, and for a sufficiently small step size $\Delta t>0$, there exists a unique solution $u^{\ell+1}$ satisfying (9).

Proof. Firstly, we will prove the coercivity of the bilinear form $B(u^{\ell+1}, v)$ by using the properties of the fractional diffusion operator and Fractional Poincaré–Friedrichs inequality,

$$\begin{aligned} B(u^{\ell+1}, u^{\ell+1}) &= (u^{\ell+1}, u^{\ell+1}) + \Delta t({}_0D_L^{\gamma}u^{\ell+1}, D_x(\lambda(x)u^{\ell+1})) \\ &\ge \|u^{\ell+1}\|^2_{L^2(0,L)} + \Delta t\,\eta_2 |u^{\ell+1}|^2_{H^{(\gamma+1)/2}(0,L)} \\ &\ge C\|u^{\ell+1}\|^2_{H^{(\gamma+1)/2}(0,L)}, \end{aligned}$$

then bilinear form $B(\cdot,\cdot)$ is coercive over $H_0^{(\gamma+1)/2}(0,L)$.
Next, we will prove the continuity of the bilinear form $B(\cdot,\cdot)$ over $H_0^{(\gamma+1)/2}(0,L) \times H_0^{(\gamma+1)/2}(0,L)$ as follows:

$$\begin{aligned} |B(u^{\ell+1}, v)| &= |(u^{\ell+1}, v) + \Delta t({}_0D_L^{\gamma}u^{\ell+1}, D_x(\lambda(x)v))| \\ &\le \|u^{\ell+1}\|_{L^2(0,L)} \|v\|_{L^2(0,L)} \\ &\quad + \Delta t\,\eta_2 \|u^{\ell+1}\|_{H^{(\gamma+1)/2}(0,L)} \|v\|_{H^{(\gamma+1)/2}(0,L)} \\ &\le \widetilde{C}\|u^{\ell+1}\|_{H^{(\gamma+1)/2}(0,L)} \|v\|_{H^{(\gamma+1)/2}(0,L)}. \end{aligned}$$

Moreover, we can also prove the continuity of $F(\cdot)$ over $H_0^{(\gamma+1)/2}(0,L)$ as follows:

$$\begin{aligned} |F(v)| = |(g,v)| &\le \|g\|_{L^2(0,L)} \|v\|_{L^2(0,L)} \\ &\le \widehat{C}\|g\|_{H^{(\gamma+1)/2}(0,L)} \|v\|_{H^{(\gamma+1)/2}(0,L)}. \end{aligned}$$

Therefore, the hypotheses of Lax-Milgram theorem are satisfied [14] and then there exist a unique solution for the semi-discrete weak form (9). □

Theorem 3 (Stability of the semi-discrete problem).

For $0<\eta_1<\lambda(x)<\eta_2$, and for a sufficiently small step size $\Delta t>0$, the problem (9) is stable, and it holds

$$\|u^{\ell+1}\|_{H^{(\gamma+1)/2}(0,L)} \le -\left(\|u^0\|_{L^2(0,L)} + \Delta t \sum_{j=0}^{\ell} \|f^{\ell+1}\|_{L^2(0,L)}\right).$$

Proof. For $\ell = 0$ and $v = u^1$ then problem (8) will be

$$\left(u^1, u^1\right) + \Delta t\left(({}_0D_L^{\gamma}u^1), D_x(\lambda(x)u^1)\right) = \left(u^0, v\right) + \Delta t\left(f^1, u^1\right). \tag{10}$$

The right hand side of (10) will be

$$\left(u^1,u^1\right)+\Delta t\left(({}_0D_L^{\gamma}u^1),D_x(\lambda(x)u^1)\right)$$
$$\geq C_1\left\|u^1\right\|^2_{H_0^{(\gamma+1)/2}(0,L)}. \quad (11)$$

The left hand side of (10)

$$\left(u^0,u^1\right)+\Delta t\left(f^1,u^1\right)$$
$$\leq\left\|u^0\right\|_{L^2(0,L)}\left\|u^1\right\|_{L^2(0,L)}+\Delta t\left\|f^1\right\|_{L^2(0,L)}\left\|u^1\right\|_{L^2(0,L)}$$

(From Lemma 1)

$$=\left(\left\|u^0\right\|_{L^2(0,L)}+\Delta t\left\|f^1\right\|_{L^2(0,L)}\right)\left\|u^1\right\|_{L^2(0,L)}$$
$$\leq C_2\left(\left\|u^0\right\|_{L^2(0,L)}+\Delta t\left\|f^1\right\|_{L^2(0,L)}\right)\left\|u^1\right\|_{H_0^{(1+\gamma)/2}(0,L)}. \quad (12)$$

From (11) and (12) we have

$$\left\|u^1\right\|_{H_0^{(\gamma+1)/2}(0,L)}\leq\frac{C_2}{C_1}\left(\left\|u^0\right\|_{L^2(0,L)}+\Delta t\left\|f^1\right\|_{L^2(0,L)}\right). \quad (13)$$

For $\ell\geq 1$ so we have

$$\left\|u^{\ell}\right\|_{H_0^{(\gamma+1)/2}(0,L)}\leq C_3\left(\left\|u^{\ell-1}\right\|_{L^2(0,L)}+\Delta t\left\|f^{\ell}\right\|_{L^2(0,L)}\right). \quad (14)$$

From (13), (14) we obtain

$$\left\|u^{\ell+1}\right\|_{H_0^{(\gamma+1)/2}(0,L)}\leq C_4\left(\left\|u^{\ell}\right\|_{L^2(0,L)}+\Delta t\sum_{j=0}^{\ell}\left\|f^{\ell}\right\|_{L^2(0,L)}\right). \quad \square$$

Now, El-gendi nodal Galerkin method discretization proceeds by approximating the solution the polynomials of high degree. So we introduce a finite dimensional space $P_0^N=P^N\cap H_0^{(\gamma+1)/2}(0,L)$ where P^N is the space of all polynomials in which the polynomial degree is less than or equal to N and the space is given as follows $P_0^N=span\{\varphi_1(x),\varphi_2(x),...,\varphi_{N-1}\}$,

where $\varphi_j(x)$ are given by:

$$\varphi_j(x)=\frac{2\theta_j}{N}\sum_{k=0}^{N}\theta_k T_k((2/L)x_j-1)T_k((2/L)x-1),$$
$$j=0,1,...,N, \quad (15)$$

for all $\theta_k=1$, except $\theta_0=\theta_N=1/2$ and

$$\varphi_j(x_k)=\begin{cases}0 & if \quad j\neq k,\\ 1 & if \quad j=k.\end{cases}$$

The grid points x_k are the extrema points of the shifted Chebyshev polynomial $T_k((2/L)x-1)$. Let the approximate solution is given as follows

$$u(x,t)\approx u^N(x,t)=\sum_{i=0}^{N}U_i(t)\phi_i(x), \quad (16)$$

to ensure the approximations satisfy the boundary conditions, we set $U_0=U_N=0$. Also, since the test function $v(x)$ as a function of Nth order polynomials so we can write these polynomials in the equivalent cardinal form

$$v(x)=\sum_{l=0}^{N}V_l\,\varphi_l(x),$$

where the nodal values V_l are arbitrary, except that $V_0=V_N=0$ to ensure that v satisfies the boundary conditions. Now the discrete weak form is given as follows: find $u^N\in P_0^N$

$$\left(u_t^N,v\right)_N=-\left(({}_0D_L^{\gamma}u_N),D_x(\lambda)v)\right)_N+\left(f_N,v\right)_N,$$
$$\forall v\in P_0^N,\,t>0, \quad (17)$$

where the inner product $(g,h)_N$ is evaluated as follows

$$(g,h)_N=\sum_{j=0}^{N}\tilde{b}_{Nj}g(x_j)h(x_j),$$

and

$$x_i=\frac{L}{2}(y_i+1),\quad i=0,...,N,$$
$$y_j=\cos\left(\frac{j\pi}{N}\right),\quad j=0,1,...,N.$$

The quantities b_{Nj} are given by: [9]

$$b_{Nj}=\frac{4}{N}\sum_{i=0}^{N/2}\frac{\theta_s}{4i^2-1}\cos\frac{2j\pi i}{N},\qquad j=1,2,...,N-1,$$
$$b_{N0}=b_{NN}=\frac{1}{N^2-1}. \quad (18)$$

Since $0\leq x\leq L$ then the mapped weights will be given from the following relation $\tilde{b}_{Nj}=\frac{2}{L}b_{Nj}$. Then the first discrete inner product becomes

$$\left(u_t^N,v\right)_N=\sum_{j=0}^{N}\omega_j\left(\sum_{n=0}^{N}\dot{U}_n\varphi_n(x_j)\sum_{m=0}^{N}V_m\varphi_m(x_j)\right),$$

since $\varphi_i(x_j)=\delta_{ij}$, then the sum reduces to

$$\left(u_t^N,v\right)_N=\sum_{j=0}^{N}\tilde{b}_{Nj}\dot{U}_jV_j, \quad (19)$$

where

$$\dot{U}_j=\frac{dU_j}{dt}.$$

For evaluating the second term in (17), let

$${}_0D_L^{\gamma}u^N=D^{\gamma}u^N=\sum_{l=0}^{N}U_l(t)\varphi_l^{\gamma}(x),\quad 0<\gamma<1,$$

then the fractional derivative of the cardinal function can be written as:

$$\varphi_l^{\gamma}(x)=\frac{2\theta_l}{N}\sum_{k=0}^{N}\theta_k T_k((2/L)x_j-1)D^{\gamma}T_k((2/L)x-1),$$
$$l=0,1,...,N,$$

where

$$x_i=\frac{L}{2}(y_i+1),\; y_i=-\cos\left(\frac{i\pi}{N}\right)\quad i=0,...,N,$$

and the Caputo fractional derivative of the Shifted Chebyshev polynomial is:

$$D^{\gamma}T_k((2/L)x-1)=\frac{1}{\Gamma(1-\gamma)}\int_0^x\frac{T_k'((2/L)t-1)}{(x-t)^{\gamma}}dt$$

for $0<\gamma<1$, where the derivatives of Chebyshev polynomial T_i satisfy

$$T_0 = T_1',\ T_1 = \frac{T_2'}{4}, \ldots,\ T_i = \frac{T_{i+1}'}{2(i+1)} - \frac{T_{i-1}'}{2(i-1)},\ i \ge 2,$$

so, we can deduce that the recurrence relations

$$\frac{T_0'}{2} = 0,\ T_i' = 2i(T_{i-1} + T_{i-3} + \ldots + T_1), \quad i \text{ even}, \tag{20}$$

$$T_i' = 2i(T_{i-1} + T_{i-3} + \ldots + 0.5T_0), \quad i \text{ odd}. \tag{21}$$

Then from eq. (20-21) we can deduce the original modal differentiation matrix $\tilde{D}$ in the spectral space. $\tilde{D}$ is a sparse upper triangular matrix with interties

$$\tilde{d}_{ij} = \begin{cases} 0, & if \quad i = j, \\ 0, & if\ (j-i)\ even, \\ 2j, & otherwise. \end{cases}$$

Then the Caputo fractional derivative of the Shifted Chebyshev polynomial is [4]and given as follows:

$$M_0(x) = \frac{x^{1-\gamma}}{2(1-\gamma)},\ M_1(x) = \frac{(2/L)x^{2-\gamma}}{(1-\gamma)(2-\gamma)} - \frac{x^{1-\gamma}}{(1-\gamma)}, \tag{23}$$

$$M_2(x) = \frac{4(2/L)^2 x^{3-\gamma}}{(3-\gamma)(2-\gamma)(1-\gamma)} - \frac{4(2/L)x^{2-\gamma}}{(1-\gamma)(2-\gamma)} + \frac{x^{1-\gamma}}{(1-\gamma)}, \tag{24}$$

and for $n = 3,4,\ldots,N$ we have the following recurrence relation

$$\left(1+\frac{1-\gamma}{k}\right)(M_k(x)) = 2((2/L)x-1)(M_{k-1}(x))^{1-\gamma} + \left(-1+\frac{1-\gamma}{k-2}\right)(M_{k-2}(x)) - \left(\frac{2(-1)^k}{k(k-2)}\right)x \tag{25}$$

hence, by substituting (23), (24) and (25) in (22), then we have

$$D^\gamma T_k((2/L)x-1) = \frac{1}{\Gamma(1-\gamma)} \sum_{n=0}^{N} \tilde{d}_{kn} M_n(x). \tag{26}$$

Consequently, the fractional derivative of the cardinal function is given in the following form:

$$D^\gamma \varphi_l(x) = \frac{2\theta_l}{N\Gamma(1-\gamma)} \sum_{k=0}^{N}\sum_{n=0}^{N} \theta_k T_k((2/L)x_j - 1)\tilde{d}_{kn} M_n(x),$$

$$l = 0,1,\ldots,N,\ 0 \le x \le L.$$

The second term in (11) can be evaluated as follows:

$$\left((D^\gamma u^N),(\lambda)v)'\right)_N = \sum_{j=0}^{N} V_j \Big(\sum_{k=0}^{N} \tilde{b}_{Nk}[D^\gamma u^N(x)(\lambda(x)\varphi_j'(x) + \lambda'(x)\varphi_j(x))]\Big)\Big|_{x=x_k}, \tag{27}$$

where the first derivative of the cardinal functions $\varphi_j(x)$ at the points x_k is derived in [11]. Similarly, the source term is given as follows:

$$\left(f^N, v\right)_N = \sum_{j=0}^{N} \tilde{b}_{Nj} f_j^N V_j. \tag{28}$$

From equations (19), (27) and (28) we have

$$\sum_{j=0}^{N} V_j (\tilde{b}_{Nj}\dot{U}_j + \sum_{k=0}^{N} \tilde{b}_{Nk}[D^\gamma u^N(x)[\lambda(x)\varphi_j'(x) + \lambda'(x)\varphi_j(x)]]\Big|_{x=x_k} - \tilde{b}_{Nj} f_j^N = 0.$$

Since V_j 's are linearly independent, the coefficient of each V_j must be zero, so

$$\tilde{b}_{Nj}\dot{U}_j = -\sum_{k=0}^{N} \tilde{b}_{Nk}[D^\gamma u^N(x)(\lambda(x)\varphi_j'(x) + \lambda'(x)\varphi_j(x))]\Big|_{x=x_k} + \tilde{b}_{Nj} f_j^N,\ j = 1,\ldots,N-1,$$

and

$$\tilde{b}_{Nj}\dot{U}_j = -\sum_{l=0}^{N} (B_{jl} + C_{jl})U_l + \tilde{b}_{Nj} f_j^N, \quad j = 1,\ldots,N-1,$$

$$U_0 = U_N = 0, \tag{29}$$

where

$$B_{jl} = \sum_{k=0}^{N} \tilde{b}_{Nk}\lambda(x_k)\varphi_l^\gamma(x_k)\varphi_j'(x_k),\ C_{jl} = \tilde{b}_{Nj}\lambda'(x_j)\varphi_l^\gamma(x_j).$$

Let $A_{jl} = -(B_{jl} + C_{jl})$, then (29) can be written as

$$\tilde{b}_{Nj}\dot{U}_j = \sum_{l=0}^{N} A_{jl} U_l + \tilde{b}_{Nj} f_j^N, \quad j = 1,\ldots,N-1,$$

with the boundary conditions

$$U_0 = U_N = 0.$$

Then the fully discrete problem is given in the following form

$$\tilde{b}_{Nj}\left(\frac{U_j^{\ell+1} - U_j^{\ell}}{\Delta t}\right) = \sum_{l=0}^{N} A_{jl} U_l^{\ell+1} + \tilde{b}_{Nj} f_j^{\ell+1},$$

$$j = 1,\ldots,N-1,\ \ell = 1,2,\ldots,$$

with the boundary conditions

$$U_0^{\ell+1} = U_N^{\ell+1} = 0. \tag{30}$$

3.2 STABILITY FOR FULL DISCRETE PROBLEM

In this section we use recall Leray-Schauder fixed point theorem [18] to prove the existence of the solution for eq. (30).

Lemma 3. For a given open and bounded domain $\Omega \subset R^n$ containing the origin $0 \in \Omega$ and let $E : \Omega \to R^n$ be a continuous function. If $E(x) \ne \mu x$ for all $\mu > 1$ and $x \in \Omega$ then E has a fixed point in $\bar{\Omega}$ which is the closure of Ω.

We introduce discrete norm which induced from the discrete inner product

$$\| g \|_N = \sqrt{(g,g)_N} = \left(\sum_{j=0}^{N} \tilde{b}_{Nj} g(x_j) g(x_j)\right)^{1/2}.$$

Theorem 4. The equation (30) has a solution.

Proof. Let $\Omega = B(0,a) \subset R^n$ be a ball centered at the origin $0 \in \Omega$ with radius a and consider $\partial\Omega$ be the boundary of Ω. Let $U = (U_0,\ldots,U_N)$ be a vector on the boundary $\partial\Omega$ such that for some $\varepsilon > 1$

$$\varepsilon U = E(U) + U^{\ell}, \tag{31}$$

where

$$E_N(U) = \frac{\Delta t}{\tilde{b}_{Nj}} \sum_{l=0}^{N} A_{jl} U_l^{\ell+1} + \Delta t\, f_j^{\ell+1},$$

by taking a discrete inner product of (30) with U then we have

$$\varepsilon \|U\|^2_{L^2(0,L)} = (E(U), U) + (U^{\ell}, U)$$

$$= \frac{\Delta t}{\tilde{b}_{Nj}} (AU, U) + \Delta t (f, U) + (U^{\ell}, U),$$

from [2] and since $f(x,t)$ is continuous on $[0, L] \times [0, T]$ then by Gronwall Lemma we have: $\| f \| \le \kappa,\ \kappa > 0$ and by applying Caushy-Schwarz inequality we have

$$\varepsilon \|U\|_N^2 \le \frac{\Delta t}{\tilde{b}_{Nj}} \|A\| \|U\|_N^2 + \kappa \Delta t \|U\|_N^2 + \|U^{\ell}\|_N \|U\|_N, \tag{32}$$

where $\|A\|$ is bounded by a positive number. Divide both sides of (32) by $\|U\|_N^2$ then we have

$$\varepsilon \le \frac{\Delta t}{\tilde{b}_{Nj}} \|A\| + \kappa \Delta t + \frac{\|U^{\ell}\|_N}{\|U\|_N},$$

then for large $\|U\|_N$ and for very small Δt then $\varepsilon \le 1$ which implies to contradiction. So

$$\varepsilon U \ne E(U) + U^{\ell}, \qquad \forall \varepsilon > 1, \quad U \in \partial\Omega.$$

Hence there is a solution $U \in \partial\Omega$ such that $U = E(U) + U^{\ell}$. □

Theorem 5. For any fixed N, the full discrete scheme (30) is stable.

Proof. From equation (30) and by taking a discrete inner product with $U^{\ell+1}$ and since $\| f \|_N \le \kappa,\ \kappa > 0$. Then, from Caushy-Schwarz inequality we have

$$\|U^{\ell+1}\|_N^2 - \|U^{\ell}\|_N^2 \le \frac{\Delta t}{\tilde{b}_{Nj}} \|A\| \|U^{\ell+1}\|_N^2 + \kappa \Delta t \|U^{\ell+1}\|_N^2, \tag{33}$$

then by summing (33) from $\ell = 0$ to $\ell = M$, we obtain

$$\|U^M\|_N^2 \le \|U^0\|_N^2 + \left(\frac{\Delta t}{\tilde{b}_{Nj}} \|A\| + \kappa \Delta t \right) \sum_{\ell=0}^{M} \|U^{\ell+1}\|_N^2,$$

by the discrete Gronwall inequality, we obtain

$$\|U^M\|_N \le \exp\left(\frac{CT}{2}\right) \left[\|U^0\|_N^2 + \left(\frac{\Delta t}{\tilde{b}_{Nj}} \|A\| + \kappa \Delta t \right) \times \sum_{\ell=0}^{M} \|U^{\ell+1}\|_N^2 \right]^{\frac{1}{2}}. \qquad \square$$

3.3 NONLINEAR CASE

In this section we will illustrate how we can use the nodal Chebyshev Galerkin method to solve the nonlinear diffusion equation. So, we will give the weak form of problem (3) and (4) in case $a = 0, b = L$ is given as follows: Find $u \in H_0^{(\gamma+1)/2}(0, L)$ such that:

$$(w_t, v) = -\left((a(w)\,{}_0D_x^{\gamma} w), D_x v\right) - \left(w, D_x(\rho v)\right) + \left(f(x,t,w), v\right), \forall v \in H_0^{(\gamma+1)/2}(0, L),\ t > 0.$$

The existence and uniqueness of the weak form is proved in [23]. Let the approximate solution is given as follows:

$$w(x,t) \approx w^N(x,t) = \sum_{i=0}^{N} W_i(t) \phi_i(x),$$

so the above weak form can be written as

$$\left(w_t^N, v\right)_N = -\left((a(w^N)\,{}_0D_x^{\gamma} w^N), D_x v\right)_N - \left(w^N, D_x(\rho v)\right)_N + \left(f(x,t,w^N), v\right)_N, \forall v \in P_0^N, t > 0. \tag{34}$$

After some manipulations we have

$$\tilde{b}_{Nj} \dot{W}_j = -\sum_{l=0}^{N} \overline{B}_{jl} W_l - \sum_{l=0}^{N} \overline{C}_{jl} W_l - \sum_{l=0}^{N} \overline{H}_{jl} W_l + f(x_j, t, w_j^N),\ \ j = 1, ..., N-1, \tag{35}$$

where

$$\overline{B}_{jl} = \sum_{k=0}^{N} \tilde{b}_{Nk} a(w_k^N) \varphi_l^{\gamma}(x_k) \varphi_j'(x_k),$$

$$\overline{C}_{jl} = \tilde{b}_{Nl}\, \rho(x_l) \varphi_j'(x_l),$$

$$\overline{H}_{jl} = \tilde{b}_{Nl}\, \varphi_j'(x_l) \rho(x_l),$$

and hence (35) is given as

$$\tilde{b}_{Nj} \dot{W}_j = -\sum_{l=0}^{N} \overline{B}_{jl} W_l - \sum_{l=0}^{N} \overline{A}_{jl} W_l + f(x_j, t, w_j^N),\ \ j = 1, ..., N-1, \tag{36}$$

where

$$\overline{A}_{jl} = \overline{C}_{jl} + \overline{H}_{jl},$$

To approximate the time derivative, we will use the backward Euler finite difference for the linear parts while the forward Euler finite difference for the nonlinear part. Let f_k^{ℓ} is the approximation of $f(x_k, t_{\ell})$. Then (36) is approximated by

$$\tilde{b}_{Nj} W_j^{\ell+1} + \Delta t \sum_{l=0}^{N} \overline{A}_{jl} W_l^{\ell+1} = \tilde{b}_{Nj} W_j^{\ell} - \Delta t \sum_{l=0}^{N} \left[\sum_{k=0}^{N} \tilde{b}_{Nk} a(W_k^{\ell}) \varphi_l^{\gamma} \varphi_j'(x_k)\right] + f_j^{\ell}(W_j^{\ell}),\ \ j = 1, ..., N-1, \ell = 1, 2, ...,$$

with the boundary conditions

$$W_0^{\ell+1} = W_N^{\ell+1} = 0. \tag{37}$$

4. NUMERICAL EXPERIAMENTS

In this section we will give numerical examples and we will use MATLAB 8 software to obtain the numerical results.

Example 1: Consider the following space fractional order differential equation:

$$u_t - (\Gamma(1.2)x^{1.8})D_x^{1.8}u = f(x,t), \;\; 0 \le x \le 1, \; 0 \le t \le 1,$$

where $f(x,t) = (6x^3 - 3x^2)e^{-t}$ with the initial and boundary conditions:

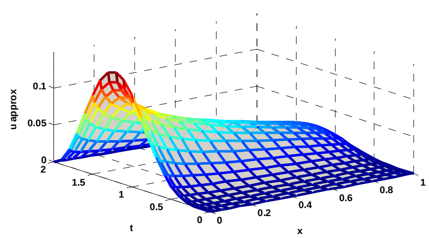
$$u(x,0) = (x^2 - x^3), \qquad 0 \le x \le 1,$$

$$u(0,t) = 0, \; u(1,t) = 0, \qquad 0 \le t \le T .$$

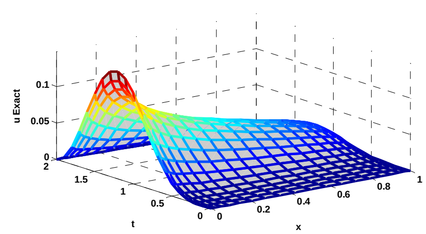
The exact solution is $u(x,t) = (x^2 - x^3)e^{-t}$. The numerical results are shown in table 1 and figures 1. In table 1, we give the absolute errors between the exact solution $u(x,t)$ and the approximate solution $u^N(x,t)$ and we make a comparison with results obtained by method in [12] at the interior points at final time $T = 2$ with time step $\Delta t = 0.0025$.

TABLE 1: The absolute error between the exact and approximate solutions in the interior points at $T = 2$.

X	Nodal Method	Method [12]
0.1	5.33 e-06	4.20 e-05
0.2	8.26 e-06	3.76 e-05
0.3	8.85 e-06	8.44 e-05
0.4	8.34 e-06	3.27 e-05
0.5	7.45 e-06	3.61 e-05
0.6	6.32 e-06	1.94 e-05
0.7	5.07 e-06	2.95 e-05
0.8	2.71 e-06	4.92 e-05
0.9	9.81 e-07	2.83 e-05



(a)



(b)

Fig.1: (a) plot of the approximate solution, (b) plot of the exact solution for $N = 30$ and $\Delta t = 0.1$.

It is noted from Table 1 and Fig. 1that we can achieve a good approximation for the exact solution by using El-gendi Galerkin method and also our results are in good agreement with the method introduced in [12].

Example 2: Consider the following nonlinear space fractional order differential equation [23]:

$$w_t = D_x(w^2 D_x^{0.5}w) - d(x,t)D_x w - w - f(x,t)w^2, \qquad 0 \le x \le 1, \; 0 \le t \le 1,$$

where

$$d(x,t) = 2e^{-2t}x(x-1)\left[\frac{2x^{1.5}}{\Gamma(2.5)} - \frac{x^{0.5}}{\Gamma(1.5)}\right],$$

$$f(x,t) = e^{-t}\left[\frac{2x^{0.5}}{\Gamma(1.5)} - \frac{x^{-0.5}}{\Gamma(0.5)}\right],$$

with the initial and boundary conditions:

$$w(x,0) = x(x-1), \qquad 0 \le x \le 1,$$

$$w(0,t) = w(1,t) = 0, \qquad 0 \le t \le 1.$$

In this case the exact solution is $u(x,t) = e^{-t}x(x-1)$. The numerical results are shown in table 2 and figure 2. In table 2, we give the maximum error between the exact solution $w(x,t)$ and the approximate solution $w_N(x,t)$, in the interior points with different time steps. Note that the maximum error is defined as follows:

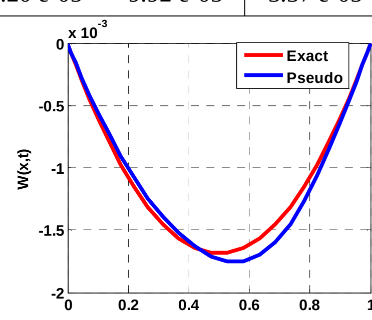
$$\|w - w_N\|_\infty = \max_{x_i}\left(\left|w(x_i,t) - w_N(x_i,t)\right|\right), \; i = 1,\ldots, N-1.$$

TABLE 2: The maximum error for $N = 10$

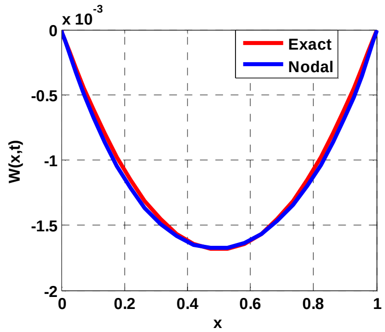
Δt	$T = 1$	$T = 3$	$T = 5$	$T = 10$
10^{-1}	5.60 e-03	1.80 e-03	3.90 e-04	6.22 e-06
10^{-2}	2.60 e-03	4.67 e-04	7.09 e-05	6.19 e-07
10^{-3}	2.40 e-03	3.83 e-04	5.28 e-05	3.68 e-07
10^{-4}	2.40 e-03	3.75 e-04	5.10 e-05	3.45 e-07

TABLE 3: The comparison between El-gendi nodal Galerkin and pseudo-spectral methods for different N and time $T = 5$.

Δt	$N = 20$		$N = 30$	
	Nodal Method	Pseudo Method	Nodal Method	Pseudo Method
10^{-1}	3.94 e-04	5.53 e-04	3.95 e-04	5.54 e-04
10^{-2}	7.05 e-05	1.37 e-04	7.12 e-05	1.42 e-04
10^{-3}	5.36 e-05	1.01 e-04	5.50 e-05	1.06 e-04
10^{-4}	5.20 e-05	9.92 e-05	3.37 e-05	1.03 e-04



(a)



(b)

Fig 2. (a) Pseudo Method at $T = 5$, $\Delta t = 10^{-2}$ and $N = 30$, (b) El-gendi Nodal method at $T = 5$, $\Delta t = 10^{-2}$ and $N = 30$.

It is clear from table 2, when the time step be smaller; we obtain a good accuracy although for long time. On the other hand, in table 3, we make a comparison between the nodal method and the pseudo-spectral method for constant final time and for different number of grid points at different time steps. We note that at the time step $\Delta t = 10^{-4}$ and for $N = 20$ the maximum error of the nodal method is (5.20 e-05). Moreover, when the number of grid points increased ($N = 30$) the maximum error decrease to reach (3.37e-05). However, pseudo-spectral method at the same time step and when the number of grid points increased the maximum error increased from (9.92 e-05) to (1.03 e-04). Also, we can observe that from Figure 2 which ensures our numerical results. So, our method is convergent and stable in the numerical sense.

5. CONCLUSION

In this article, we propose a new technique for solving linear and nonlinear fractional advection-diffusion equation numerically. The method based on the Chebyshev polynomial and the fractional derivatives are described in the Caputo sense. The solution obtained using the proposed method shows that this approach can solve the problem effectively. Comparisons are made between the approximate and exact solutions illustrate the validity and the great potential of the proposed technique.

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