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OBSERVATIONS ON THE HYPERBOLOID OF TWO SHEETS $7X^2 -3Y^2 = Z^2 + Z(Y-X) + 4$.

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Abstract - ***Infinitely many non-zero distinct integral points on the hyperboloid of two sheets given by $7x^2- 3y^2 = z^2 + z(y-x) + 4$ are obtained. A few interesting properties among the solutions are presented.***

Keyword - Ternary quadratic equation, Hyperboloid of two sheets, Integral points, MSC 2000 Mathematics subject classification: 11D09.

NOTATIONS

- $T_{m,n}$ – Polygonal number of rank n with size m
- P_n^m - Pyramidal number of rank n with size m

INTRODUCTION

The ternary quadratic Diophantine equations offer an unlimited field for research because of their variety [1,2]. For an extensive review of various problems one may refer [3-11]. In this context, one may also refer [12-18].This communication concerns with yet another interesting ternary quadratic equation $7x^2- 3y^2 = z^2 + z(y-x) + 4$ representing hyperboloid of two sheets for determining its infinitely many non zero integral solutions. Also a few interesting relations among the solutions are presented. Employing the integral solution on the given hyperboloid of two sheets, a few fascinating relations among special, polygonal and pyramidal numbers are given..

METHOD OF ANALYSIS

The ternary quadratic equation representing the hyperboloid of two sheets is

$$7x^2- 3y^2 = z^2 + z(y-x) + 4 \quad (1)$$

Introduction of the linear transformations

$$x = X + 3T, \quad y = X + 7T, \quad z = 2T. \quad (2)$$

in equation (1) leads to

$$X^2 = 24T^2 + 1$$

which is the well known Pellian equation whose general solution (X_n ,T_n) is given by,

$$X_n = \frac{1}{2}\left[\left(5+2\sqrt{6}\right)^{n+1}+\left(5-2\sqrt{6}\right)^{n+1}\right]$$

$$T_n = \frac{1}{4\sqrt{6}}\left[\left(5+2\sqrt{6}\right)^{n+1}-\left(5-2\sqrt{6}\right)^{n+1}\right] \quad (3)$$

In view of (2), the non zero integral solutions of (1) are given by

$$x_n = \frac{f}{2}+\frac{3g}{2\sqrt{24}}, \quad y_n = \frac{f}{2}+\frac{7g}{2\sqrt{24}}, \quad z_n = \frac{g}{\sqrt{24}}$$

where

$$f = \left(5+2\sqrt{6}\right)^{n+1}+\left(5-2\sqrt{6}\right)^{n+1}$$

$$g = \left(5+2\sqrt{6}\right)^{n+1}-\left(5-2\sqrt{6}\right)^{n+1}$$

The recurrence relations satisfied by x_n, y_n, z_n are correspondingly exhibited below:

$$\left.\begin{array}{lll} x_n -10x_{n+1} + x_{n+2} = 0 & x_0 = 8, & x_1=79 \\ y_n - 10y_{n+1} + y_{n+2} = 0 & y_0 = 12, & y_1= 119 \\ z_n - 10z_{n+1} + z_{n+2} = 0 & z_0 = 2, & z_1 = 20. \end{array}\right\} \quad (4)$$

A few interesting relations observed among the solutions (7) are presented below:

(i) $2x_{n+1} - 16x_n = 15z_n$
(ii) $x_{n+2} - 79x_n = 75z_n$
(iii) $y_n - x_n = 2z_n$
(iv) $2y_{n+1} - 24y_n = -25z_n$
(v) $y_{n+2} - 119y_n = -125z_n$

REMARKABLE OBSERVATIONS

Let (x,y,z) be any non-zero distinct integral solution of (1).

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1) Each of the following two triples also satisfies (1)

   (i) (-55x+36y+2z, -84x+55y+3z, z)

   (ii) (-16x + 3y +5z, y, -51x +9y + 16z)

2) A few relations noticed among the special polygonal and pyramidal numbers are presented below:

(i)

$$7\left(P_x^5\right)^2\left(T_{3,y+1}^2\right)\left(T_{3,z-2}^2\right)-3\left(3P_y^3\right)^2\left(T_{3,x}^2\right)\left(T_{3,z-2}^2\right)-\left(3P_{z-2}^3\right)^2\left(T_{3,x}^2\right)\left(T_{3,y+1}^2\right)$$

$$-\left(3P_{z-2}^3\right)\left(3P_y^3\right)\left(T_{3,x}^2\right)\left(T_{3,y+1}\right)\left(T_{3,z-2}\right)+\left(3P_{z-2}^3\right)\left(P_x^5\right)\left(T_{3,y+1}^2\right)\left(T_{3,x}\right)\left(T_{3,z-2}\right)$$

$$=4\left(T_{3,x}^2\right)\left(T_{3,y+1}^2\right)\left(T_{3,z-2}^2\right).$$

(ii)

$$7\left(P_x^3\right)^2\left(T_{3,y-2}^2\right)\left(T_{3,z}^2\right)-3\left(3P_{y-2}^2\right)^2\left(T_{3,x+1}^2\right)\left(T_{3,z}^2\right)-\left(P_z^5\right)^2\left(T_{3,x+1}^2\right)\left(T_{3,y-2}^2\right)-$$

$$\left(P_z^5\right)\left(3P_{y-2}^3\right)\left(T_{3,x+1}^2\right)\left(T_{3,y-2}\right)\left(T_{3,z}\right)+\left(P_z^5\right)\left(3P_x^3\right)\left(T_{3,y-2}^2\right)\left(T_{z,3}\right)\left(T_{3,x+1}\right)$$

$$=4\left(T_{3,x+1}^2\right)\left(T_{3,y-2}^2\right)\left(T_{3,z}^2\right).$$

3) Each of the following is a perfect square:

(i)

$$7\left(3P_{x-2}^3\right)^2\left(T_{3,y}^2\right)\left(T_{3,z+1}^2\right)-3\left(P_y^5\right)^2\left(T_{3,x-2}^2\right)\left(T_{3,z+1}^2\right)-\left(3P_z^3\right)^2\left(T_{3,x-2}^2\right)\left(T_{3,y}^2\right)$$

$$-\left(3P_z^3\right)\left(P_y^5\right)\left(T_{3,x-2}^2\right)\left(T_{3,y}^2\right)\left(T_{3,z+1}\right)+\left(3P_z^3\right)\left(3P_{x-2}^3\right)\left(T_{3,y}^2\right)\left(T_{3,x-2}\right)\left(T_{3,z+1}\right)$$

(ii)

$$7\left(P_x^3\right)^2\left(T_{3,y}^2\right)\left(T_{3,z-2}^2\right)-3\left(P_y^5\right)^2\left(T_{3,x+1}^2\right)\left(T_{3,z-2}^2\right)-\left(3P_{z-2}^3\right)^2\left(T_{3,x+1}^2\right)\left(T_{3,y}^2\right)-$$

$$\left(3P_{z-2}^3\right)\left(P_y^5\right)\left(T_{3,x+1}^2\right)\left(T_{3,y}\right)\left(T_{3,z-2}\right)+\left(P_x^3\right)\left(3P_{z-2}^3\right)\left(T_{3,y}^2\right)\left(T_{3,x+1}\right)\left(T_{3,z-2}\right)$$

4) Each of the following is congruent to zero modulo 4

(i)

$$7\left(3P_{x-2}^3\right)^2\left(T_{3,y+1}^2\right)\left(T_{3,z}^2\right)-3\left(3P_y^3\right)^2\left(T_{3,x-2}^2\right)\left(T_{3,z}^2\right)-\left(P_z^5\right)^2\left(T_{3,x-2}^2\right)\left(T_{3,y+1}^2\right)$$

$$-\left(P_z^5\right)\left(3P_y^3\right)\left(T_{3,x-2}^2\right)\left(T_{3,y+1}\right)\left(T_{3,z}\right)+\left(P_z^5\right)\left(3P_{x-2}^3\right)\left(T_{3,y+1}^2\right)\left(T_{3,x-2}\right)\left(T_{3,z}\right)$$

(ii)

$$7\left(P_x^5\right)^2\left(T_{3,y-2}^2\right)\left(T_{3,z-1}^2\right)-3\left(3P_{y-2}^3\right)^2\left(T_{3,x}^2\right)\left(T_{3,z+1}^2\right)-\left(3P_z^3\right)^2\left(T_{3,x}^2\right)\left(T_{3,y-2}^2\right)$$

$$-\left(3P_z^3\right)\left(P_{y-2}^3\right)\left(T_{3,x}^2\right)\left(T_{3,y-2}\right)\left(T_{3,z+1}\right)+\left(3P_z^3\right)\left(P_x^5\right)\left(T_{3,y-2}^2\right)\left(T_{3,x}\right)\left(T_{3,z+1}\right)$$

APPLICATION

Let (x, y, z) be any non zero positive integer solution of (1). Let m, n, r, s be any four non-zero positive integers. Choose r and s such that r>s and r-s = z. Treat r, s as the generators of the Pythagorean triangle (X,Y,W), where X=2rs, $Y=r^2 - s^2$, $W=r^2 + s^2$. Then, the relation between the solution of (1), the sides of Pythagorean triangle and special numbers is given by

$$(W-X)\left(T_{3,y+1}^2\right)\left(T_{3,x}^2\right)+\sqrt{W-X}\left(T_{3,x}\right)\left(T_{3,y+1}\right)\left[\left(3P_y^3\right)T_{3,x}-\left(P_x^5\right)T_{3,y+1}\right]$$

$$=7\left(P_x^5\right)^2\left(T_{3,y+1}^2\right)-27\left(P_y^3\right)^2\left(T_{3,x}^2\right)-4\left(T_{3,y+1}\right)\left(T_{3,x}\right)$$

In addition to the above solution pattern, we have three more patterns which are presented below:

Pattern:I

$$24x_n = 12f + 3\sqrt{6}\,g$$

$$24y_n = 12f - 7\sqrt{6}\,g$$

$$24z_n = -2\sqrt{6}\,g$$

Pattern:II

$$24x_n = 12F - 3\sqrt{6}\,G$$

$$24y_n = 12F - 7\sqrt{6}\,G$$

$$24z_n = 2\sqrt{6}\,G$$

Where $F=\left(9+4\sqrt{5}\right)^{n+1}+\left(9-4\sqrt{5}\right)^{n+1}$

$$G=\left(9+4\sqrt{5}\right)^{n+1}-\left(9-4\sqrt{5}\right)^{n+1}$$

Pattern:III

$$24x_n = 12F + 3\sqrt{6}\,G$$

$$24y_n = 12F + 7\sqrt{6}\,G$$

$$24z_n = -2\sqrt{6}\,G$$

CONCLUSION

To conclude one may search for other pattern of solutions and their corresponding properties.

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