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AN UNIFIED DEFINITION FOR ANTI-INTEGRAL AND ANTI-DERIVATIVE OPERATORS FOR ANY ORDER

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Abstract *- **Our study is based on operators approach. First, positive integer s order fractional integral is defined by iterating s-times one order integral using Euler's gamma functions properties. Then, the definition is extended to any value of s (positive , negative, fractional, transcendental, real, complex numbers) using the extension of Euler's gamma and beta functions. Many properties (limits, linearity, semi-group) are given. The most important and useful property is the semi-group one Then , the definition of fractional derivative is derived from this property. There are two types of fractional derivatives, left handed and right handed ones. The left handed fractional derivative applied to a constant function gives a non null result whereas the right handed one leads to a null result. The latter one is then better than the first one. In a previous work, we have studied the case of fractional operators applied to the set of causal functions. In the present one, we look for the set of anti-causal functions. We introduce anti-integral and anti-derivatives operators. Though properties obtained for the cases of causal and anti- causal functions are similar, there are some differences. We obtain formulae given by many authors (Liouville, Riemann, Caputo, Liouville-Caputo) as particular cases of ours.***

Keyword *-* Operators, Fractional anti-integrals, Fractional anti-derivatives, Gamma functions, Beta functions

I. INTRODUCTION

In the previous paper refered [1], we have proposed to tackle the problem of s-order fractional integrals operators J^s and s-order fractional derivatives operators D^s acting over the set E of causal functions f of variable x the following strategy:

1) define $J^1(f)(x)$ and $(f)(x)$,

2) define, from $J^1(f)(x)$ and $D^1(f)(x)$, $J^s(f)(x)$ and $D^s(f)(x)$ for any $s \in \mathbb{N}$,

3) extend to $s \in \mathbb{N}$, then to $s \in \mathbb{R}$,

4) extend to $s \in \mathbb{C}$,

5) and finally, look for in which case the results satisfy the following relations :

5a) principle of correspondence

$$\lim_{s\to n} D^s(f)(x) = \frac{d^n}{dx^n}(f) \quad for\ n \in \mathbb{N}$$

$$\lim_{s\to n} J^s(f)(x) = \int_a^x \int_a^{t_1} \int_a^{t_2} \dots \int_a^{t_n} f(t_n) dt_n dt_{n-1} \dots dt_1$$

$for\ n \in \mathbb{N}$

5b) linearity property of D^s and J^s

$$D^s(cf)(x) = cD^s(f)(x)$$

$$D^s(f_1 + f_2)(x) = D^s(f_1)(x) + D^s(f_2)(x)$$

or equivalently

$$D^s(c^1 f_1 + c^2 f_2)(x) = c^1 D^s(f_1)(x) + c^2 D^s(f_2)(x)$$

for any constants c^1 et c^2 and for any causal functions f_1 and f_2 (the same relations for $J^s(f)(x)$)

5c) the value of the fractional derivative operator D^s applied on constant function is null.

5d) semi-group property

$$J^{s_1}(J^{s_2})(f)(x) = J^{s_1+s_2}(f)(x) = J^{s_2}(J^{s_1})(f)(x)$$

$$D^{s_1}(D^{s_2})(f)(x) = D^{s_1-s_2}(f)(x) = D^{s_2}(D^{s_1})(f)(x)$$

or

$$J^{s_1}J^{s_2} = J^{s_1+s_2} = J^{s_2}J^{s_1}$$

$$D^{s_1}D^{s_2} = D^{s_1+s_2} = D^{s_2}D^{s_1}$$

Several approaches have been done [2],[3],[4], but the present strategy seems to be better and more systematic.
Let us recall some of our results [1]. We have applied the operators on the set E of causal functions f such $f(x) = 0$ for $x \leq a$ .We have shown that $J^s(f)(x)$ for $s \in \mathbb{N}$ is obtained by iterating s-times the integral

$$J^1(f)(x) = \int_a^x f(y)\,dy \qquad (1.1)$$

then

$$J^s(f)(x) = \frac{1}{\Gamma(s)} \int_a^x (x-y)^{s-1} f(y)\,dy \qquad (1.2)$$

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where $\Gamma(s)$ is Euler's gamma function for s.We have extended the definition of J^s for s being any positive fractional number, positive transcendental number $(\pi, e, \dots)$, any positive real number , any complex number with the condition real part $Re(s) > 0$ .We have studied many properties of the operator J^s in particular the most important, interesting and useful semi-group property:

$$J^{s_1}J^{s_2} = J^{s_1+s_2} = J^{s_2}J^{s_1} \qquad (1.3)$$

As $J^1D^1 = D^1J^1 = 1_E$ where 1_E is the identity operator over E because $f(a) = 0$ then

$$D^1 = J^{-1} \quad \text{and} \quad D^s = J^{-s} \quad \text{for } s \in \mathbb{N}$$

From the expression (1.2) of $J^s(f)(x)$, we have derived the expression of the s-order right handed derivative D_r^s

$$D_r^s = J^{k-s}J^{-k} = J^{k-s}D^k \text{ for } k \geq s, \ k \in \mathbb{N}, \ k \geq 1$$

because $J^{-k} = D^k$ for $k \in \mathbb{N}, k \geq 1$

or

$$D_r^s(f)(x) = \frac{1}{\Gamma(k-s)}\int_a^x (x-y)^{k-s-1} f^{(k)}(y)\,dy \qquad (1.4)$$

where

$$f^{(k)}(y) = D^{(k)}f(y) = \frac{d^k}{dx^k}(f)(y)$$

Because of the semi-group property of J^s , D_r^s is independent on the positive integer k. Choosing $k = 1$, we have

$$D_r^s = J^{1-s}D^1 \quad \text{for } 0 \leq s < 1 \qquad (1.5)$$

or

$$D_r^s(f(x) = \frac{1}{\Gamma(1-s)}\int_a^x (x-y)^{-s} f^{(1)}(y)\,dy \qquad (1.6)$$

According to the value $a = 0$ or $= +\infty$, we obtain as particular cases of our definitions (1.2) and (1.6) different expressions given by many authors (Liouville, Riemann, Caputo, Liouville-Caputo) [8].

In the present work, instead of causal functions *f*, we look for what is to be changed in the case of anticausal functions *g* of *x* i.e. $g(x) = 0$ for $x \geq b$. We will apply the same strategy. Though we have similar results, it is interesting to look for the differences. The study may be simplified by introducing the s-order anti-integral operator $\mathbb{J}^s$ and the right- handed anti-derivative $\mathbb{D}_r^s$

$$\mathbb{J}^1(g)(x) = \int_x^b g(y)\,dy \qquad (1.7)$$

$$\mathbb{J}^s(g)(x) = \frac{1}{\Gamma(s)}\int_x^b (y-x)^{s-1} g(y)\,dy \qquad (1.8)$$

$$\mathbb{D}_r^s - \mathbb{J}^{k-s}\mathbb{J}^{-k} \quad k \in \mathbb{N} \quad k \geq s \qquad (1.9)$$

$$= \mathbb{J}^{k-s}\mathbb{D}^k \qquad (1.10)$$

$$\mathbb{D}_r^s(g)(x) = \frac{1}{\Gamma(k-s)}\int_x^b (y-x)^{k-s-1}\mathbb{D}^k(g)(y)\,dy \qquad (1.11)$$

where

$$\mathbb{D}^k(g)(y) = (-1)^k\frac{d^k}{dy^k}g(y)(y) = (-1)^k g^{(k)}(y) \qquad (1.12)$$

We choose $k = 1$, then

$$\mathbb{D}_r^s = \mathbb{J}^{1-s}\mathbb{D}^1$$

or

$$\mathbb{D}_r^s(g)(x) = \frac{-1}{\Gamma(1-s)}\int_x^b (y-x)^{-s} g^{(1)}(y)\,dy \qquad (1.13)$$

We give many properties of $\mathbb{J}^s$ and $\mathbb{D}_r^s$. And we show that the expression (1.8) is valid for any value of s (positive, negative, integer , real and complex).For positive real s or positive real part of s if s is complex, $\mathbb{J}^s$ corresponds to fractional anti-integral $\mathbb{J}^s$ and for negative real s or negative real part of s if s is complex, $\mathbb{J}^s$ corresponds to fractional right handed antiderivative $\mathbb{D}_r^s$.

All along our paper, we adopt the operators product notation

$$ABC(g) = A[B[C(g)]]$$

II. DEFINITION OF ONE ORDER INTEGRAL OPERATOR, THE ONE ORDER NORMAL DERIVATIVE AND THE ONE ORDER ANTIDERIVATIVE

Let V be the set of integrable and derivable anticausal function g defined on the interval $I =]-\infty, b[, b \in \mathbb{R}$ such

$$g(x) = 0 \text{ for } x \geq b$$

We may consider $g(b)$ different from zero too.

We define the one order anti-integral operator $\mathbb{J}^1$, the one order anti-derivative operator $\mathbb{D}^1$, and the one order normal derivative D^1 by the relations

$$\mathbb{J}^1(g)(x) = \int_x^b g(t)\,dt \qquad (2.1)$$

$$\mathbb{D}^1(g)(x) = -\frac{d}{dx}(g)(x) = -g^{(1)}(x) \qquad (2.2)$$

$g^{(1)}$ being the first derivative of g.

$$\mathbb{D}^1 = -D^1 = -\frac{d^1}{dx^1} \qquad (2.3)$$

It is interesting to introduce the one order anti-derivative $\mathbb{D}^1$ instead of the one order normal derivative D^1. The prefix"anti" is proposed to take into account of the sign minus. We will see in the section III the reason.

The term anti-derivative is sometimes used to design the primitive or the indefinite integral of a function. We propose then to exclude this sense of anti-derivative.

III. RELATION BETWEEN AND OPERATORS

The following relations may be shown easily

$$\mathbb{D}^1\mathbb{J}^1 g(x) = -D^1\int_x^b g(t)dt = \mathbb{D}^1[G(b) - G(x)] = g(x) \quad (3.1)$$

$G(x)$ is a primitive of $g(x)$.

$\mathbb{D}^1\mathbb{J}^1 = 1_V$ where 1_V is the identity operator over V.

If $g(b) = 0$,

$$\mathbb{J}^1\,\mathbb{D}^1(g)(x) = \mathbb{J}^1\,(\mathbb{D}^1 g)(x) = -\int_x^b g^{(1)}(t)dt$$

$$= g(x) - g(b) = g(x)$$

$$\mathbb{J}^1\mathbb{D}^1 = \mathbb{D}^1\mathbb{J}^1 = 1_V \quad (3.3a)$$

If $g(b) \neq 0$

$$\mathbb{J}^1\mathbb{D}^1 \neq 1_V \quad (3.3b)$$

From the theory of the inverse of an operator, any operator A has ***at least*** a right handed inverse A_r such as

$$AA_r = 1_{Val(A)} \quad (3.4)$$

where $Val(A)$ is the value domain of A ; A_r depends on a choice [5], [6]

By an appropriate mechanism, A_r may be chosen to be unique in order that A_r is an operator.
Taking account of the relation $(3.3a)$ and $(3.3b)$ and (3.4) we may define for $A \equiv \mathbb{J}^1$ the right handed inverse $\mathbb{D}_r^1$ of $\mathbb{J}^1$

$$\mathbb{J}^1\mathbb{D}_r^1 = 1_{Val(\mathbb{J}^1)} \quad (3.5)$$

Remarks

$a)$ The condition $g(b) = 0$ depends on two parameters: the function g and the choice of the upper bound b in the anti-integral $\mathbb{J}^1$ (2.1)

For instance,

if $g(x) = e^{-x}$, b may be taken equal to $+\infty$, $g(+\infty) = 0$
if $g(x) = e^x - 1$, b may be taken equal to 0 , $g(0) = 0$

$b)$We assume $g(b) = 0$ then the anti-derivative operator $\mathbb{D}^1$ is the inverse $\mathbb{J}^{-1}$ of the one order anti-integral operator $\mathbb{J}^1$.

$$\mathbb{D}^1\mathbb{J}^1 = \mathbb{J}^1\mathbb{D}^1 = 1_V \quad (3.6)$$

IV. DEFINITIONS OF FOR ANY VALUE OF THE NUMBER

For any positive integer s, we define the integral operator $\mathbb{J}^s$ by iterating s-times $\mathbb{J}^1(g)(x)$

$\mathbb{J}^0(g)(x) = g(x)$ or $J^0 = 1_V$

$$\mathbb{J}^1(f)(x) = \int_x^b g(t)dt$$

$$\mathbb{J}^2(g)(x) = \int_x^b [\int_{t_1}^b g(t_2)\,dt_2]\,dt_1$$

...

$$\mathbb{J}^s(g)(x) = \int_x^b \mathbb{J}^{s-1} g(t)dt$$

$$= \int_x^b\int_{t_1}^b\int_{t_2}^b \dots \int_{t_{s-1}}^b g(t_s)\,dt_s dt_{s-1}\dots dt_2 dt_1 \quad (4.1)$$

It may be shown (see for instance [6])

$$\mathbb{J}^s(g)(x) = \frac{1}{\Gamma(s)}\int_x^b (y-x)^{s-1}\,g(y)dy \quad (4.2)$$

$$= \frac{x^s}{\Gamma(s)}\int_1^{b/x} (u-1)^{s-1}\,g(ux)du \quad (4.3)$$

where $\Gamma(s)$ is the Euler's gamma function for s. We obtain the relation (4.3) from (4.2) by the change of variable $u = y/x$.

We may extend the definition of $\mathbb{J}^s$ for any $s \in \mathbb{R}_+$ [6] and for any complex number $s \in \mathbb{C}$ with the condition $Re(s) > 0$ [7]

V. SOME PROPERTIES OF

Theorem 1

$\mathbb{J}^s$ is a linear operator over V.

Proof

$$\mathbb{J}^s(\lambda_1 g_1 + \lambda_2 g_2)(x) = \lambda_1\mathbb{J}^s(g_1)(x) + \lambda_2 J^s(g_2)(x) \quad (5.1)$$

for g_1 and g_2 belonging to V and for any complex λ_1 and λ_2.
The proof is very easy.

Theorem 2

We have the semi-group property of J^s

$$\mathbb{J}^{s_1}\mathbb{J}^{s_2} = \mathbb{J}^{s_1+s_2} = \mathbb{J}^{s_2}\mathbb{J}^{s_1} \quad (5.2)$$

Proof
We have to show

$$\mathbb{J}^{s_1}\mathbb{J}^{s_2}(g)(x) = \mathbb{J}^{s_1+s_2}(g)(x)$$

for any $g \in V$ and for any s_1 and s_2, real positive integers, real fractional numbers, real numbers , complex numbers with $Re(s_1) > 0$ and $Re(s_2) > 0$.
First, let us suppose s_1 and s_2 real positive integer numbers

$$\mathbb{J}^{s_1}\mathbb{J}^{s_2}(g)(x) = \frac{1}{\Gamma(s_1)\Gamma(s_2)}\int_x^b (y-x)^{s_1-1}dy\int_y^b (z-y)^{s_2-1}g(z)dz$$

We apply the Dirichlet's formula given by Whittaker and Watson [8] ,[9]

$$\int_x^b dy(y-x)^{\alpha-1}\int_y^b dz\,(z-y)^{\beta-1}dy\gamma(y,z)$$

$$= \int_z^b dz\int_x^z dy(y-z)^{\alpha-1}(z-y)^{\beta-1}\gamma(y,z) \qquad (5.3)$$

for $\alpha = s_1, \beta = s_2, \gamma(y,z) = g(z)$

$$\mathbb{J}^{s_1}\mathbb{J}^{s_2}(g)(x) = \frac{1}{\Gamma(s_1)\Gamma(s_2)}\int_a^x dz\,g(z)\int_x^z dy(y-x)^{s_1-1}(z-y)^{s_2-1}$$

Then, we change the variable y in the second integral into u

$$u = \frac{z-y}{z-x}$$

or

$$y = z - u(z-x)$$

$$dy = -\,du(z-x)$$

$$\int_x^z dy(y-x)^{s_1-1}(z-y)^{s_2-1}$$

$$= (z-x)^{s_1+s_2-1}\int_0^1 du(1-u)^{s_1-1}u^{s_2-1}$$

$$= (z-x)^{s_1+s_2-1}B(s_1)(s_2)$$

where B is the Euler's beta function

$$B(s_1)(s_2) = \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2)}$$

then

$$\mathbb{J}^{s_1}\mathbb{J}^{s_2}(g)(x) = \frac{1}{\Gamma(s_1+s_2)}\int_x^b dz g(z)(z-x)^{s_1+s_2-1}$$

$$= \mathbb{J}^{s_1+s_2}(g)(x)$$

As $J^{s_1+s_2}$ is symmetric in s_1 and s_2, we have

$$\mathbb{J}^{s_1}\mathbb{J}^{s_2}(g)(x) = \mathbb{J}^{s_1+s_2}(g)(x) = \mathbb{J}^{s_2}\mathbb{J}^{s_1}(g)(x)$$

or

$$\mathbb{J}^{s_1}\mathbb{J}^{s_2} = \mathbb{J}^{s_1+s_2} = \mathbb{J}^{s_2}\mathbb{J}^{s_1}$$

We have supposed that s_1 and s_2 are positive integer numbers. But if we use the extensions of Euler's gamma and beta functions, it may be easily shown that the semi-group property of $\mathbb{J}^s$ stands true for any s_1 and s_2 (positive fractional, positive real,) [6] and for any s_1 and s_2 complex numbers with the conditions $Re(s_1) > 0$ and $Re(s_2) > 0$ [7].

This important property is very useful to shorten the demonstration of many interesting formulae, in particular the semi-group property of s-order anti-derivative $\mathbb{D}^s$. (see section VII)

VI. DEFINITION OF THE -ORDER ANTI-DERIVATIVE OPERATOR DERIVED FROM

We deduce the definition of the $s-$order anti-derivative operator $\mathbb{D}^s$ from the definitions of s-order integral operator $\mathbb{J}^s$.

For this purpose, we take advantage of the relation (3.3) or (3.6)

$$\mathbb{J}^1\mathbb{D}^1 = \mathbb{D}^1\mathbb{J}^1 = 1_V$$

$\mathbb{D}^1$ is the inverse operator $\mathbb{J}^{-1}$ of $\mathbb{J}^1$ (6.1)

$\mathbb{J}^1$ is the inverse operator $\mathbb{D}^{-1}$ of $\mathbb{D}^1$ (6.2)

By mathematical induction, for s positive integer, we may define

$$\mathbb{D}^s = \mathbb{J}^{-s} \qquad (6.3)$$

$$\mathbb{J}^s = \mathbb{D}^{-s} \qquad (6.4)$$

Then we may extend the definition of $\mathbb{J}^s$ (see formula (4.2)) given for s positive integer into s negative integer for any real (positive or negative number) and for any complex number by using the semi-group property of $\mathbb{J}^s$.
For the definition of anti-derivative $\mathbb{D}^s$ for $s > 0$, we have **two possibilities :** left- handed anti-derivative $\mathbb{D}_l^s$ and right-handed anti-derivative $\mathbb{D}_r^s$:

a) $\mathbb{D}_l^s = \mathbb{J}^{-k}\mathbb{J}^{k-s} = \mathbb{D}^k\mathbb{J}^{k-s}$ for $k \in \mathbb{N}, k \geq s, s \geq 1$ (6.5)

$$\mathbb{D}_l^s(g)(x) = \mathbb{D}^k\mathbb{J}^{k-s}(g)(x) \qquad (6.6)$$

$$= \mathbb{D}^k\frac{1}{\Gamma(k-s)}\int_a^x (y-x)^{k-s-1}g(y)dy \qquad (6.7)$$

$$= \mathbb{D}^1\frac{1}{\Gamma(1-s)}\int_a^x (y-x)^{-s}g(y)dy \qquad (6.8)$$

with the choice $k-1$ in the relation (6.7).

$\mathbb{D}_l^s$ ***is independent on the value of*** $\boldsymbol{k}$ because of the semi-group property of $\mathbb{J}^s$.
If $g(x) = C$ where C is any constant number ,then

$$\mathbb{D}_l^s(C) = \frac{C(b-x)^{-s}}{\Gamma(1-s)} \neq 0 \quad \text{if } C \neq 0 \qquad (6.9)$$

The s order left handed anti-derivative of a non null constant is not null, but depends on x, which is an absurd result. We have ***to disregard this possibility*** because it is ***in contradiction*** with the correspondence principle 5c).

b) $\mathbb{D}_r^s = \mathbb{J}^{k-s}\mathbb{J}^{-k} = \mathbb{J}^{k-s}\mathbb{D}^k$ for $k \in \mathbb{N}, k \geq s, k \geq 1$ $\quad (6.10)$

$$\mathbb{D}_r^s(g)(x) = \mathbb{J}^{k-s}(\mathbb{D}^k g)(x) \qquad (6.11)$$

$$= \frac{(-1)^k}{\Gamma(k-s)} \int_x^b (y-x)^{k-s-1} g^{(k)}(y)\,dy \qquad (6.12)$$

$$= \frac{-1}{\Gamma(1-s)} \int_x^b (y-x)^{-s} g^{(1)}(y)\,dy \qquad (6.13)$$

with the choice $k = 1$.

$\mathbb{D}_r^s$ ***is independent on*** $\boldsymbol{k}$ because of the semi-group property of $\mathbb{J}^s$.
If $g(x) = C$ where C is a constant, then

$\mathbb{D}_r^s(C) = 0$ even if the constant $C \neq 0$ $\quad (6.14)$

The choice $\mathbb{D}_r^s$ in the relation (6.10) is better than $\mathbb{D}_l^s$ in the relation (6.5) because of the relation (6.14) instead of the relation (6.9) which does not verify the condition: the fractional derivative of any constant is equal to zero.

c) If $0 < s < 1$, for the sake of simplicity, we may choose $k = 1$

$$\mathbb{D}_r^s(g)(x) = \mathbb{J}^{1-s}\mathbb{D}^1 g(x) \qquad (6.15)$$

We apply the derivative under the integration sign and we integrate by part after, the terms $(y-x)^{-s}g(y)$ to be taken for $y = x$ minus one term for $y = b$ cancels each other without introducing any condition on the function g. Finally we obtain

$$\mathbb{D}_r^s(g)(x) = \frac{1}{\Gamma(1-s)} \int_x^b (y-x)^{-s}\,\mathbb{D}^1 g(y)\,dy \quad (6.16)$$

$$= \frac{-1}{\Gamma(1-s)} \int_x^b (y-x)^{-s} g^{(1)}(y)\,dy \quad (6.17)$$

because $\mathbb{D}^1 g(y) = -\frac{d^1}{dx^1}(g)(y) = -g^{(1)}(y)$
If we take $b = +\infty$, the expression (6.17) is the Liouville-Caputo fractional derivative definition [8]:

$$_{LC}D_-^s(g)(x) = {}_L\mathbb{J}_-^{1-s}\frac{d}{dx}(g)(x) \qquad (6.18)$$

apart from the sign minus.
The lower script LC means Liouville-Caputo; the lower scripts L and $-$ in ${}_LJ_-^{1-s}$ mean respectively Liouville for Liouville fractional integral definition, the integral being “right handed integral”; it collects weighted function values for $x > y$ which means right hand side from x. If y is time-like coordinate, the right handed integral corresponds to an “anticausal” function g. In our notations, the relation (6.10) for $k = 1$ is

$$\mathbb{D}_r^s(g)(x) = \mathbb{J}^{1-s}\mathbb{D}^1(g)(x) \qquad (6.19)$$

d) If $s > 1$, we put $s = E(s) + s_1$ where $E(s)$ is the entire part of s and $0 \leq s_1 < 1$.

$$\mathbb{D}_r^s = \mathbb{D}^{E(s)+s_1}$$
$$= \mathbb{D}^{s_1}\mathbb{D}^{E(s)} \text{ (semi-group property of } \mathbb{D}^s\text{)}$$

It means that ***the fractional right handed anti-derivative*** $\mathbb{D}_r^s$ ***is an integer order anti-derivative*** $\mathbb{D}^{E(s)}$ ***followed by a fractional antiderivative*** $\mathbb{D}^{s_1}$ ***with*** $\mathbf{0 \leq s_1 < 1}$.

We recall

$$\mathbb{D}^n = (-1)^n \frac{d^n}{dx^n} \quad \text{for} \quad n \in \mathbb{N}$$

e) If we take $k = 2, 3, 4, \ldots$ in the relation (6.10) we will obtain the same result. Direct calculations are not difficult but tedious.

VII. DUAL PROPERTIES OF FRACTIONAL ANTIDERIVATIVES

It is easy to derive from properties of fractional integrals $\mathbb{J}^s$ dual properties for $\mathbb{D}^s$, in particular the two following ones :
- linear property

$$\mathbb{D}^s(\lambda_1 g_1 + \lambda_2 g_2)(x) = \lambda_1\mathbb{D}^s(g_1)(x) + \lambda_2\mathbb{D}^s(g_2)(x) \quad (7.1)$$

for $\lambda_1, \lambda_2 \in \mathbb{C}$ and $g_1, g_2 \in E$
- semi-group property

$$\mathbb{D}^{s_1}\mathbb{D}^{s_2} = \mathbb{D}^{s_1+s_2} = \mathbb{D}^{s_2}\mathbb{D}^{s_1} \qquad (7.2)$$

The direct proofs are not difficult, they may be performed by a similar way as for fractional integral calculations [1]

VIII. LIMITS FOR FRACTIONAL ANTI-INTEGRALS OPERATORS AND FRACTIONAL ANTI-DERIVATIVES OPERATORS

Let us now look for the correspondence principle 5a) in the Introduction. We have to show the following limits

$\lim_{s \to n} \mathbb{J}^s = \mathbb{J}^n \quad (s \in \mathbb{R}_+, n \in \mathbb{N})$

$\lim_{s \to n} \mathbb{D}^s = \mathbb{D}^n \quad (s \in \mathbb{R}_+, n \in \mathbb{N})$

where $\mathbb{J}^n$ and $\mathbb{D}^n$ are respectively integer order anti- integrals and integer order antiderivatives operators.

Theorem 3

First, let us assume $0 \leq s \leq 1$

$$\lim_{s\to 0^+} \mathbb{J}^s = 1_V$$

$$\lim_{s\to 0^+} \mathbb{D}^s = 1_V$$

or

$$\lim_{s\to 0^+} \mathbb{J}^s = \mathbb{J}^0 = 1_V$$

$$\lim_{s\to 0^+} \mathbb{D}^s = \mathbb{D}^0 = 1_V$$

Proof

$$\mathbb{J}^s(g)(x) = \frac{1}{\Gamma(s)}\int_x^b (y-x)^{s-1}\,\mathbb{D}^1 g(y)\,dy \qquad (8.1)$$

$$= \frac{-1}{\Gamma(s+1)}\int_x^b (y-x)^s\, g^{(1)}(y)\,dy \qquad (8.2)$$

where

$$g^{(1)}(y) = \frac{d^1}{dy^1}g(y)$$

For $s = \varepsilon \ll 1$

$$J^\varepsilon(g)(x) = \frac{-1}{\Gamma(1+\varepsilon)}\int_x^b (y-x)^\varepsilon g^{(1)}(y)\,dy$$

We give the Taylor's series expansion of

$$\frac{1}{\Gamma(1+\varepsilon)}(y-x)^\varepsilon - 1 + \varepsilon[\gamma + ln(y-x)]$$
$$+\frac{\varepsilon^2}{2}[\gamma^2 - \frac{\pi^2}{6} + 2\gamma\, ln(y-x) + ln^2(y-x)] + \cdots \quad (8.3)$$

where $\gamma = 0.577\,215\,665\,901\,532\,860\,6\ldots$ is the Euler – Mascheroni's constant [8]

$$\mathbb{J}^0(g)(x) = -\int_x^b g^{(1)}(y)\,dy \qquad (8.4)$$

$$= g(x) \qquad (8.5)$$

because $g(b) = 0$

or $\mathbb{J}^0 = 1_V$

By duality, we have the same relation for $\mathbb{D}^0$.

Theorem 4

We have the dual relations for $\mathbb{J}^s$ and $\mathbb{D}^s$, $s \in \mathbb{C}$ with $Re(s) > 0$.

Proof

The proof is obtained easily in copying the demonstration given in the causal function case [1].

Theorem 5

For instance, let us verify directly the relations

$$\lim_{\varepsilon\to 0^+} \mathbb{D}^{1-\varepsilon} g(x) = -\frac{d^1}{dx^1}g(x) \qquad (8.7)$$

Proof

We start from the expression (6.15) of $\mathbb{D}_r^\varepsilon(g)(x)$ for $s = 1-\varepsilon$ with $\varepsilon > 0$ and $\varepsilon \ll 1$

$$\mathbb{D}^{1-\varepsilon}(g)(x) = \frac{1}{\Gamma(\varepsilon)}\int_x^b (y-x)^{\varepsilon-1}\frac{dg}{dx}dx \qquad (8.8)$$

We integrate first by parts

$$\mathbb{D}^{1-\varepsilon}(g)(x) - \frac{1}{\Gamma(\varepsilon)}\{-[\frac{(y-x)^\varepsilon}{\varepsilon}g^{(1)}(y)]_{y=a}^{y=x}$$
$$+\int_x^b \frac{(y-x)^\varepsilon}{\varepsilon}g^{(2)}(y)\,dy\}$$

We integrate by parts again the integral containing $g^{(2)}$. The integrated part cancels exactly the first term in the right hand side of the relation.

$$\mathbb{D}^{1-\varepsilon}(g)(x) = \frac{1}{\Gamma(\varepsilon)}\int_x^b \frac{(y-x)^\varepsilon}{\varepsilon}g^{(2)}(y)dy \qquad (8.10)$$

$$= \frac{1}{\Gamma(1+\varepsilon)}\int_x^b (y-x)^\varepsilon g^{(2)}(y)dy \qquad (8.11)$$

We utilise the Taylor's series expansion (8.3)

$$\lim_{\varepsilon\to 0^+} \mathbb{D}^{1-\varepsilon}(g)(x) = \int_x^b g^{(2)}(y)\,dy$$
$$= g^{(1)}(b) - g^{(1)}(x)$$
$$= -\frac{d^1}{dx^1}g(x) \quad \text{if and only if } g^{(1)}(b) = 0$$

Theorem 6

$$\lim_{\varepsilon\to 0^+} \mathbb{J}^\varepsilon(g)(x) = -\int_x^b g^{(1)}(y)dy$$
$$= g(x) - g(b)$$
$$= g(x) \quad \text{if and only if } g(b) = 0$$

$$\lim_{\varepsilon\to 0^+} \mathbb{J}^\varepsilon = 1_V \qquad \lim_{\varepsilon\to 0^-} \mathbb{J}^\varepsilon = 1_V$$

Theorem 7

$$\lim_{\varepsilon\to 0^+} \mathbb{J}^{1-\varepsilon} = J^1 \qquad (8.12)$$

Proof

$\lim_{\varepsilon\to 0^+} \mathbb{J}^{1-\varepsilon} = \mathbb{J}^1 \quad \lim_{\varepsilon\to 0^+} \mathbb{J}^{-\varepsilon} = \mathbb{J}^1$

because of semi-group property and

$\lim_{\varepsilon\to 0^+} \mathbb{J}^{-\varepsilon} = 1$

The direct calculation is not difficult.

$$\mathbb{J}^{1-\varepsilon}(g)(x) = \frac{1}{\Gamma(1-\varepsilon)}\int_x^b (y-x)^{-\varepsilon}\, g(y)dy$$

We utilize the Taylor's series expansion (8.3) by changing ε into $-\varepsilon$

$$\lim_{\varepsilon\to 0^+} \mathbb{J}^{1-\varepsilon}(g)(x) = \int_x^b g(y)\,dy = \mathbb{J}^1(g)(x)$$

$$\lim_{\varepsilon\to 0^+} \mathbb{J}^{1-s} = \mathbb{J}^1$$

Theorem 8

Let us assume now $s > 1$. We put $s = n + s_1$ with $n = E(s) \in \mathbb{N}$ and $0 \le s_1 < 1$.

$$\lim_{s_1\to 0^+} \mathbb{J}^{n+s_1} = \mathbb{J}^n \qquad (8.13)$$

$$\lim_{s_1\to 0^+} \mathbb{J}^{n+1-s_1} = \mathbb{J}^{n+1} \qquad (8.14)$$

Proof
The proof is immediate by utilizing the semi-group property of $\mathbb{J}^s$.

Remark
The relation (8.7) may be obtained by duality from the relation (5.2) by taking account of

$$\mathbb{D}^{1-\varepsilon} = \mathbb{J}^{\varepsilon-1+1}\mathbb{D}^1$$

$$\lim_{\varepsilon\to 0^+} \mathbb{D}^{1-\varepsilon} = \lim_{\varepsilon\to 0^+} \mathbb{J}^{\varepsilon}\mathbb{D}^1 = \mathbb{D}^1 = -\frac{d^1}{dx^1}$$

IX. COMPARISON WITH OTHER DEFINITIONS

a) If $g(b) = 0$, there is one definition for $\mathbb{J}^s$ given by the relation (4.2). But there are two possibilities for the definition of the anti-derivatives, the left handed anti-derivative $\mathbb{D}_l^s$ given by the relation (6.5) or (6.6) and the right handed anti-derivative $\mathbb{D}_r^s$ given by the relation (6.10) or (6.11).The $\mathbb{D}_l^s$ applied on a constant function is different from zero whereas the $\mathbb{D}_r^s$ applied on a constant function is null.

b) If $b = +\infty$, the relation (4.2) gives

$$\mathbb{J}^s(g)(x) = \frac{1}{\Gamma(s)}\int_x^{+\infty} (y-x)^{s-1}\, g(y)dy \qquad (9.1)$$

which is the definition of Liouville fractional integral ${}_L\mathbb{J}_-^s(g)(x)$; the under script L stands for Liouville and the under script $(-)$ is related to the right-handed integral [formula (5.16), p.36 in reference [8].

c) If $b = 0$, the relation (4.2) gives

$$\mathbb{J}^s(g)(x) = \frac{1}{\Gamma(s)}\int_x^0 (y-x)^{s-1}\, g(y)\,dy \qquad (9.2)$$

which is the Riemann fractional integral ${}_R\mathbb{J}_-^s(g)(x)$ for right handed integral , the under script R stands for Riemann and the under script (-) is related to right handed integrals (formula 5.18) p.36 in reference [8])

d) For the expression of the fractional left handed anti-derivative $\mathbb{D}_l^s$, we have

$$\mathbb{D}_l^s(g)(x) = \mathbb{D}^1\mathbb{J}^{1-s}(g)(x)$$

$$= -\frac{d}{dx}\frac{1}{\Gamma(1-s)}\int_x^b (y-x)^{-s}\, g(y)\,dy \qquad (9.3)$$

If $b = +\infty$, we obtain

$$\mathbb{D}_l^s(g)(x) = -\frac{d}{dx}\frac{1}{\Gamma(1-s)}\int_x^{+\infty} (y-x)^{-s}\, g(y)\,dy \quad (9.4)$$

which is the definition of Liouville fractional derivative ${}_L\mathbb{D}_-^s(g)(x)$ apart from the sign ; the under script L means Liouville and the under script (-) is related to right handed integral. [formula (5.31) p.38 in reference [8])

e) If $b = 0$, the relation (6.8) gives

$$\mathbb{D}_l^s(g)(x) = -\frac{d}{dx}\frac{1}{\Gamma(1-s)}\int_x^0 (y-x)^{-s}\, g(y)dy \quad (9.5)$$

which is minus the definition of Riemann fractional derivative

${}_R\mathbb{D}_-^s(g)(x)$; the under script R means Riemann and the under script (-) means right handed integral (formula (5.35) p.39 in reference [8]).

We have the $\mathbb{D}_l^s$ applied on a constant function is not null.

f) We have the relation (6.17):

$$\mathbb{D}_r^s(g)(x) = \frac{-1}{\Gamma(1-s)}\int_x^b (y-x)^{-s}\, g^{(1)}(y)\,dy \; (9.6)$$

g) If $b = 0$, the relation (9.6) is minus the definition of Caputo fractional derivative ${}_C\mathbb{D}_-^s(g)(x)$; the under script C means Caputo and the under script (-) corresponds to right handed integral (formula (5.48) p.42 in the reference [8]).
The $\mathbb{D}_r^s$ applied on a constant function is null.

h) If $b = +\infty$, the relation (9.6) is minus the definition of Liouville-Caputo fractional derivative ${}_{LC}D_-^s(g)(x)$; the under script LC means Liouville-Caputo and the under script (-) stands for right handed integral (formula (5.41) p.40 in reference [8]).

X. CONCLUSIONS

Let us conclude by stressing once more that the definition of fractional right handed (respect. left handed) anti-derivative may be obtained from the definition of an integer order ordinary derivative D^k followed (respect. preceded) by fractional anti-integral by

$$\mathbb{D}_l^s = \mathbb{D}^k \mathbb{J}^{k-s}$$

$$\mathbb{D}_r^s = \mathbb{J}^{k-s}\mathbb{D}^k \text{ for any integer } k > s$$

In the second member of these relations, we point out that there is no summation over k and it is independent on k. Comparing the expression (4.2) of $\mathbb{J}^s(g)(x)$ and $(6.3), (6.4), (6.10), (6.17)$ of $\mathbb{D}^s(g)(x)$, we obtain an ***unified*** definition for fractional anti-integrals and fractional anti-derivatives operators according to ***the sign of the real part*** of the order s.

It is worth pointing out that definitions given by many authors (Liouville, Riemann, Caputo, Liouville- Caputo) apart from the sign are particular cases of our formula.

APPENDIX

Case of $g^{(m)}(b) \neq 0$ for $m = 0, 1, 2, 3, \ldots$

Let us assume $g^{(m)}(b) \neq 0$ for $m = 0, 1, 2, 3 \ldots$ and look for some consequences.

Theorem A1

$$\mathbb{J}^s(g)(x) = \mathbb{J}^{s+1}\left(-g^{(1)}\right)(x) + \frac{1}{\Gamma(s+1)}(b-x)^s g^{(0)}(b) \qquad (A.1)$$

where

$$-g^{(1)} = -\frac{d}{dx}(g) = \mathbb{D}^1(g)$$

is minus the first ordinary derivative of g.

Proof

$$\mathbb{J}^s(g)(x) = \frac{1}{\Gamma(s)}\int_x^b (y-x)^{s-1} g(y)dy$$

We integrate by parts and take account of $s\Gamma(s) = \Gamma(s+1)$

$$\mathbb{J}^s(g)(x) = \mathbb{J}^{s+1}\left(-g^{(1)}\right)(x) + \frac{(b-x)^s}{\Gamma(s+1)} g(a)$$

Theorem A2

Theorem A.1 may be generalized for any positive integer k

$$\mathbb{J}^s(g)(x) = \mathbb{J}^{s+k}[(-1)^k g^{(k)}(x)] + \sum_{m=1}^{m\le k}\frac{(b-x)^{s-m-1}}{\Gamma(s+m)}(-1)^{m-1}g^{(m-1)}(b) \qquad (A.2)$$

$$= \frac{1}{\Gamma(s+k)}\int_x^b (y-x)^{s+k-1}(-1)^k g^{(k)}(y)dy + \sum_{m=1}^{m\le k}\frac{(b-x)^{s+m-1}}{\Gamma(s+m)}(-1)^{m-1}g^{(m-1)}(b)$$

Proof

We utilize mathematical induction. The relation is true for $k = 1$. Let us suppose that it is true for k and we will show that it stands true for $k+1$.

$$\mathbb{J}^s(g)(x) = \frac{1}{\Gamma(s+k)}\int_x^b (y-x)^{s+k-1}(-1)^k g^{(k)}(y)dy + \sum_{m=1}^{m\le k}\frac{(b-x)^{s+m-1}}{\Gamma(s+m)}(-1)^{m-1}g^{(m-1)}(b)$$

We integrate by parts the integral in the right hand side

$$\frac{1}{\Gamma(s+k)}\int_x^b (y-x)^{s+k-1}(-1)^k g^{(k)}(y)dy =$$

$$\frac{1}{\Gamma(s+k)}\int_x^b \frac{(y-x)^{s+k}}{(s+k)}(-1)^{k+1}g^{(k+1)}(y)dy + \frac{1}{\Gamma(s+k)}\frac{(b-x)^{s+k}}{(s+k)}(-1)^k g^{(k)}(a)$$

Then

$$\mathbb{J}^s(g)(x) = \frac{1}{\Gamma(s+k+1)}\int_x^b (y-x)^{s+k}(-1)^{k+1}g^{(k+1)}(y)dy + \sum_{m=1}^{m\le k+1}\frac{(b-x)^{s+m-1}}{\Gamma(s+m)}(-1)^{m-1}g^{(m-1)}(b)$$

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