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DANIELL INTEGRATION FOR BANACH SPACE VALUED VECTOR MAPS

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Abstract- ***The paper introduces integration of Banach space valued mappings by using a simple construction and variant of Daniell stone integral for functions by adapting the construction of Michael Leinert with some changes.. This can give a simple treatment of the stochastic Ito integral retaining it's intuitive flavour.***

Key-Words: Bochner integral, Lebesgue integral , Daniell construction ,measure Ams subject classification (28B 1)

I. INTRODUCTION

We discuss Daniell integration of Banach space valued vector maps. Bochner integral that is Lebesgue integral for Banach space valued maps is discussed in Lang [2] including description of vector measures and Radon Nikodym theorem for vector measures. Bochner integral using Daniell like integration is discussed in the monograph of Mikisuinski[[6] but domain is Euclidean space. Essentially the same method is used in Asplund and Bungart [1] to discuss Daniell integration over an arbitrary ground set but for real valued functions.

We adapt the method employed in the paper of professor Michael Leinert [3] . However the paper discusses the integration of real valued functions only, but we discuss Daniell integral for Banach space valued vector maps defined over arbitrary ground set X. The proofs are simple and elegant . The method can give a simple description of the stochastic Ito integral while retaining its intuitive flavour.

We reserve the term function when codomian is Real numbers and whenever the codomain is possibly a Banach space or a vector space we use the term mapping.

Baisc terminology : To fix the ideas one can interpret S as the set of positive step functions in Euclidean spaces or positive continuous functions on X.

Definitions [1] :

Seminormed space : A seminormed space Y, is a vector space over $\Re$ or $\mathbb{C}$, with a nonnegative real valued function

norm denoted by $\|\ .\ \|$ defined on Y satisfying following properties

1) $\| y \| \geq 0$, for all $y \in Y$.
2) $\| 0 \| = 0$
3) $\| \lambda\, y \| = \| \lambda \| . \| y \|$,for $y \in$ and $\lambda \in F$, $F = \Re$ or $\mathbb{C}$
4) $\| x + y \| \leq \| x \| + \| y \|$, $x, y \in Y$.

A **seminormed space is called a normed space, if $\| y \| = 0$, implies $y = 0$**, for **all $y \in Y$. Clearly then, $\| x - y \|$ induces a metric on Y.**

A normed space or a seminmormed which is complete in the metric induced by the norm is called a **Banach space**, that is every Cauchy sequence is convergent .

Examples:

1) Any Euclidean space $\Re^n$ is a banach space over $\Re$.
2) $\Re$, $\mathbb{C}$ are Banach spaces over $\Re$.
3) C[a, b] = the vector space of continuous functions defined over the bounded closed interval [a, b] is a Banach space under the norm,
$|| f ||_\infty = \sup \{ | f(x) | : x \in [a, b] \}$.

Remarks : Nothing essentially is lost if the reader substitutes a Euclidean space for a Banach space.

A **Pseudonomed space** is a notion almost identical with that of a seminormed space except that the norm function $\|\ .\ \|$ is permitted to take the value □

E can be taken as space of all step mappings taking values in a Banach space Y.

$\|\ .\ \|$ denotes the function given the integral for S which can be integral for step functions, or Riemann integral for continuous functions.

I is the mapping which gives integral on E which can be integral for step mappings , or Henstock-Kurzweil integral for Henstock-Kurzweil integrable mappings .

Let X be any ground set. Let E be a vector space of maps defined over X taking values a Bancah space Y (As an exception We allow Y to be $\Re^*$, addition treated as 0 when undefined $0. \infty = 0.-\infty = 0$)| .

Let S be a positive cone formed by positive functions defined on taking values in $\Re^+$ that is closed for addition and scalar multiplication by nonnegative scalar. And the zero function belongs to S.

A Daniell configuration is a pair (E, S) with an integral operator I defined on E taking values in Y and a positive extended real valued norm function $\|\ .\ \|$ defined on S.

1) I is a linear mapping on E.
2) $|| s_1 + s_2 || = ||s_1|| + ||s_2||$
3) $|| \lambda .s || = \lambda .I(s)$, $\lambda \geq 0$.
4) If $s_1 \geq s_2$, then $||s_1|| \geq || s_2||$
5) $|| 0 || = 0$.

Remark : ∞ is not allowed as a value for I in general except when E is the space of extended real valued functions.

Let P = Set of all functions :X → [0, ∞] called as family of positive functions.

We define an extended positive functional || .|| on P as an extension of || . || defined on S.

$\| f \| = \{ \inf \sum \|f_n\| : \sum f_n \geq f , f_n \in S\}$.

With this P and S form pseudometric spaces with d(f, g) = || |f- g| || d is a semimetriic and can take value ¥ so it is pseudometric.

Remark : Daniell condition on S is assumed implicitly.

We define the extended positive functional ||. || on E as || f || =|| (| f |) ||

Clearly || . || is countably subadditive on P.

$\| \sum f_n \| \leq \sum \| f_n \|$.

|| . || is positive homogeneous on P and isotone on P, that is if f ≤ g then || f || ≤ || g ||,

$\| \lambda f | = | \lambda | \| f |$.

Clearly on P ||. || is continuous from below, that is if , $\sum f_n \geq f$, then $\sum \| f_n \| \geq \| f \|$.

Let, Let V denote the set of all mappings from X to Y, V = { f | f: X → Y}.

Let, E′ = { h ∈ E : || h || < ∞ }.

We shall impose the Daniell condition.

Daniell condition

D | I(f) | ≤ || f ||, for f ∈ E.

Remark :

1) This is the variant of standard Daniell condtion which says that if $\{f_n\}$ is adecreasing sequence of positive functions then $\| f_n \| \to 0$, where ||. || is the functional usually given by n elementary integral like the integral for step functions or Riemann integral for continuous functions. The proof of the equivalence can be found in the standard text Asplund and Bungart [1].

When one uses extended real valued positive functions the

2) In the case of vector valued mappings Daniell condition will particularly hold if

| I(f) | ≤ I (| f |) and I(| f |) = || f ||, assuming that the elementary integral I can be defined for both real valued functions and vector valued mappings which is usually the case for Riemann integral or integral of step mappings and step functions.

Illustrations :

Some practical illustrations of the theory can be considered as follows

1)Let X = ℜ. We take S to be positive extended step functions on real line . We take E to be step mappings on real line taking values in a banch space y. then our construction will yield the Bochner integral discussed in Lang [2] . If instead of the Banach space one considers extended real valued functions defined on X , one gets Lebesgue integral on real line. The proof of Daniell condition can be found in Lang [2].

2) let X = ℜ or n dimensional Euclidean space. Let S be the set of continuous mappings whose domain is a bounded closed interval (or a bounded closed n dimensional rectangle in case of n dimensions) and S be the set of positive continuous functions whose domain is a bounded closed interval on real line(a bounded closed n dimensional rectangle in case of n dimensions) the norm and elementary integral I can both be given by Riemann integral. Dini's theorem yields the Daniell condition for the norm as discussed in Taylor A.E [8].

3) one can take X = ℜ and take the elementary integral to be a variant of rieman n stiltjes integral where the Riemann sum is always evaluated at left end point and the integrator function is a Brownian motion and the convergence is convergence in mean. One gets the stochastic integral of Ito this will be discussed in a future paper

4) as discussed in the paper by cornel Leinenkugel [5]. One can obtain Henstock-kurzweil integral on real line by this construction by taking E to be the space of all those functions on real line which possess diagonal primitive(indefinite integral0 .

5) Interestingly the construction can yield a simple proof of Planacharel theorem as discussed in the paper by Leinert .[4]

6) Further method is capable of directly define integral of vector valued maps with vector measure

Remark: Thus the method in the form presented in this paper is a significant enhancement in the theory of integration and can give short and unified treatment of integration , measure and planchareal theorem and cover Bochner integral as well as Lebesgue integral and further Ito integral and Henstock-Kurzweil integral.

Definition: A mapping f or an extended real valued function f is called a null mapping (respectively null function) if || f || =0. A set A ⊂ X is called a null set if the indicator function 1_A is a null function. A property Q is said to hold almost everywhere if Q holds except on a null set.

Theorem :

(i) || g || = 0 iff g = 0 almost everywhere.

(ii) Countable union of null sets is a null set.

(iii) g = f almost everywhere iff || g || = || f |

(iv) if I(g) < ? then { x :| g(x) | = ? } is a null set

Proof :- Suppose || g || =0 let A = {x : g(x) ≠ 0} then $1_A \leq \sum_{n=1}^{\infty} |g(x)|$ so $\| 1_a \| = 0$ by isotony and countable subadditivity of the || . ||, so a is a null set.

Conversely let b ⊂ X be a null set and g9x) = 0 on B. But $| g | \leq \sum_{n=1}^{\infty} 1_B$. Thus by definition of the norm || . || on P we have || g || = 0.

ii) Let A = $\bigcup_{n=1}^{\infty} A_n$, where each A_n is a null set. Then $1_A \leq \sum_{n=1}^{\infty} 1_n$ so 1_Am is a null function.

iii) Let N be a null set. Let M be its complement.Then it suffices to show that for any mapping g the norm of the mapping g_M defined as restriction of g to M is given by

$\| g_{M|} \|$ = is equal to || g ||. This follows from he fact $|g_M| \leq |g| \leq |g_M| + \sum_{n=1}^{\infty} 1_N$.

iv) Let A = { x ; | g(x) = ∞ }. ∀ n ∈ ℵ , we have $n.1_A \leq | g |$. Thus $n.\|1_A| \leq \|g\|$, □ ∀ n. thus $\|1_a\| = 0$.

Remark : now one can identify mappings that are equal almost everywhere . So V is a pseudonormed linear space, if one defines f(x) + g(x) =0 if it is undefined or ambiguous .

We observe that || . || defined on V is countably sub additive.

We have generalized Beppo Levy theorem, for V. So V is complete under the norm || . ||

A mapping g is called integrable ,if for each $\varepsilon > 0$, there exists a mapping f in E such that

$\| g - f_\varepsilon \| \le \varepsilon$.

A mapping g in F is called absolutely integrable, if for each $\varepsilon > 0$, there exists a mapping f in E′ such that $\| g - f_\varepsilon \| \le \varepsilon$.

L^1 (Y) is the normed space formed by absolutely integrable maps. Since L^1 is the closure of E′ in the complete space F, L^1 is complete under the norm || . ||.

The extension of the linear map I on E to a norm continuous linear mapping on L^1(Y) is called the Daniell-Leinert integral. We have Beppo Levy theorem for $L^{1.}$(Y).

When $Y = \mathfrak{R}^*$ we simply write L^1.

Let $P' = \{ f \in P \mid f$ is a limit of increasing sequences of functions in S $\}$. one can prove monotone convergence theorem for P using countable subadditivity of the norm.

Using this one has the dominate convergence theorem and fatou's lemma for L1(Y) where dominance is by positive functions in P ′ exactly as proved in Lang [ ] .

We define a set as integrable if it's characteristic function is integrable with the measure as the value of the integral and a set as measurable if its intersection with each integrable set is an integrable set. The measure of a measurable set which is not an integrable set is assigned as ∞.

One can generate a countably additive measure on X as an application of monotone convergence theorem. One an obtain Fubini theorem as in Taylor [8]

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