International Journal of Latest Research in Science and Technology **ISSN (Online):2278-5299**
Volume 4, Issue 6: Page No.15-20, November-December 2015 (special Issue Paper)
http://www.mnkjournals.com/ijlrst.htm
Special Issue on International conference on Methematics Sciences-2015(MSC-2015)
Confernece Held at Sadguru Gadage Maharaj College, Karad ,Maharastra,(India)

STUDY OF BASIS AND COORDINATE VECTORS IN PRODUCT OF VECTOR SPACES

Divate B.B.
Assistant Professor, Department of Mathematics,
H. P. T. Arts and R.Y.K. Science College,
Nashik (M.S.)-422005.
(Affiliated to SavitribaiPhule Pune University, Pune)
E-mail: rykmathsbbd@gmail.com

Abstract- ***The aim of this paper is to find coordinate vector of a vector in product of vector spaces with respect to given basis of it and to study the relation of it with coordinate vectors of vectors with respect to bases of vector spaces whose product is defined.***

Keywords - Product of vector spaces, Basis, coordinate vector.

I. INTRODUCTION

In literature on linear algebra [1-5] we study concept of vector space over a field of characteristic zero. Several results about basis and its consequences had been studied. In algebra [6-8] we study groups, rings, fields and properties of product of these algebraic structures. Also, in topological spaces we study product of topological spaces, product of modules etc. Author defined the product of vector spaces and studied basis of finite dimensional vector space of product of vector spaces over a field [9].

Product of Vector Spaces:

Theorem 1: Let $V(+_V, \cdot_V)$and $W(+_W, \cdot_W)$ be vector spaces over a field$F$.

Let$V \times W = \{(v,w)/\ v\epsilon V, w\epsilon W\}$. For $(v_1,w_1),(v_2,w_2)\epsilon V \times W$and $k\epsilon F$ define vector addition $(+)$ and scalar multiplication $(\cdot)$operation on $V \times W$ as follows:

$(v_1,w_1) + (v_2,w_2) = (v_1 +_V v_2, w_1 +_W w_2)$and
$k.(v,w) = (k._V v, k._W w)$.

Then $V \times W$ is the vector space with respect to defined operations over the field F.

(This vector space is called as product of two vector spaces.)

Proof: By using definition of operations and vector axioms of vector spaces V and W over field F, the proof of theorem is straight forward.

In product vector space:

$(\bar{0}_V, \bar{0}_W)$is the zero vector whenever $\bar{0}_V$ is the zero vector in vector space V and $\bar{0}_W$ is the zero vector in vector space W .

$(-v,-w)$is the negative vector of the vector $(v,w)\epsilon\ V \times W$ whenever $-v,-w$ are the negative vectors of $v\ \epsilon\ V, w\ \epsilon\ W$ respectively.

1. Basis and Dimension:

Theorem 2: Let $V(+_V, \cdot_V)$and $W(+_W, \cdot_W)$ be m and n dimensional vector spaces respectively over a field F then $V \times W$ is $m+n$ dimensional vector space over the field F.i.e. $dim.(V \times W) = dim.V + dim.W$.

Proof: Let$B_V = \{v_1, v_2, \ldots, v_m\}$and $B_W = \{w_1, w_2, \ldots, w_n\}$ be basis for vector spaces V and W respectively. Let $\bar{0}_V, \bar{0}_W$ be zero vectors in V and W respectively. Define the set,

…..

We prove that a set B is basis for the vector space $V \times W$ which contains$m+n$ vectors.

Step I: $\boldsymbol{B}$ is linearly independent set in$\boldsymbol{V \times W}$.

For this

$a_1(v_1, \bar{0}_W) + a_2(v_2, \bar{0}_W) + \cdots + a_m(v_m, \bar{0}_W) + b_1(\bar{0}_V, w_1) + \cdots + b_n(\bar{0}_V, w_n) = (\bar{0}_V, \bar{0}_W)$

.

After simplifying and equating we get,

$a_1 \cdot_V v_1 +_V a_2 \cdot_V v_2 +_V \ldots +_V a_m \cdot_V v_m = \bar{0}_V$and
$b_1 \cdot_W w_1 +_W b_2 \cdot_W w_2 +_W \ldots +_W b_n \cdot_W w_n = \bar{0}_W$.

Since, B_V and $B_W$are basesfor vector spaces V and W respectively and linearly independent sets in vector spaces V and W respectively.

Therefore, $a_i = 0, i = 1,2,\ldots,m$and $b_j = 0, j = 1,2,\ldots,n$.

This proves $\boldsymbol{B}$ is linearly independent set in $V \times W$.

Step 2: To show that $\boldsymbol{B}$ span $\boldsymbol{V \times W}$

For this suppose $(v,w)\epsilon\ V \times W$ is expressed as

$(v,w) = a_1(v_1, \bar{0}_W) + a_2(v_2, \bar{0}_W) + \cdots + a_m(v_m, \bar{0}_W) + b_1(\bar{0}_V, w_1) + \cdots + b_n(\bar{0}_V, w_n)$

(1)After simplifying and equating we get,

$a_1 \cdot_V v_1 +_V a_2 \cdot_V v_2 +_V \ldots +_V a_m \cdot_V v_m = v$

(2)$b_1 \cdot_W w_1 +_W b_2 \cdot_W w_2 +_W \ldots +_W b_n \cdot_W w_n = w$.

(3)Since, B_V and $B_W$are bases for vector spaces V and W respectively and hence span vector spaces V and W respectively.
Therefore there exist $a_i \in F, i = 1,2,\ldots,m$ and $b_j \in F, j = 1,2,\ldots,n$ which satisfy (2), (3) and hence (1).
This proves B span vector space $V \times W$.
From Step 1 and step 2, B is a basis for the vector space $V \times W$ containing $m + n$ vectors.
Therefore,$dim.(V \times W) = dim.V + dim.W$.
Thus theorem is proved.

Theorem 3: Let $V(+_V, \cdot_V)$and $W(+_W, \cdot_W)$ be m and n dimensional vector spaces over a field F. Let $B_V = \{v_1, v_2, \ldots, v_m\}$and $B_W = \{w_1, w_2, \ldots, w_n\}$ be bases for vector spaces V and W respectively.Let $\bar{0}_V, \bar{0}_W$ be zero vectors in V and W respectively. Then for $v \in V$ and $w \in W$ the following sets,

$$B_1 = (B_V \times \{\bar{0}_W\}) \cup = \{(v_1, \bar{0}_W), (v_2, \bar{0}_W), \ldots, (v_m, \bar{0}_W), (v, w_1), (v, w_2), \ldots, (v, w_n)\}$$

And
$$B_2 = (B_V \times \{w\}) \cup (\{\bar{0}_V\} \times B_W) = \{(v_1, w), (v_2, w), \ldots, (v_m, w), (\bar{0}_V, w_1), (\bar{0}_V, w_2), \ldots, (\bar{0}_V, w_n)\}$$

are bases for the vector space $V \times W$.

Proof: For linear independence ofB_1 ,
$$a_1(v_1, \bar{0}_W) + a_2(v_2, \bar{0}_W) + \cdots + a_m(v_m, \bar{0}_W) + b_1(v, w_1) + \cdots + b_n(v, w_n) = (\bar{0}_V, \bar{0}_W)$$

Implies
$$a_1 \cdot_V v_1 +_V a_2 \cdot_V v_2 +_V \ldots +_V a_m \cdot_V v_m + (b_1 + b_2 + \cdots + b_n) \cdot_V v = \bar{0}_V$$
And $b_1 \cdot_W w_1 +_W b_2 \cdot_W w_2 +_W \ldots +_W b_n \cdot_W w_n = \bar{0}_W$.

Since, $B_W = \{w_1, w_2, \ldots, w_n\}$ is linearly independent in vector space W
$\Rightarrow b_j = 0$ for each $j = 1,2,\ldots,n$.
Substituting these values in above equation we get,
$a_1 \cdot_V v_1 +_V a_2 \cdot_V v_2 +_V \ldots +_V a_m \cdot_V v_m = \bar{0}_V$.
Since, $B_V = \{v_1, v_2, \ldots, v_m\}$ is linearly independent in vector space V
$\Rightarrow a_i = 0$ for each i$= 1,2,\ldots,m$.
Thus, a_i , $b_j = 0$, for each $i$and$j$.
Therefore the set B_1 is linearly independent in vector space $V \times W$.
By using similar argument we prove the set B_2 is linearly independent in vector space $V \times W$.
Moreover,
$n(B_1) = n(B_2) = m + n = dim.V + dim.W = dim.(V \times W)$
.

Therefore, by sufficient condition for basis of finite dimensional vector space, the sets B_1 and $B_2$are bases for the vector space $V \times W$.

1. **Coordinate vectors:**

Theorem 4: Let $B_1 = \{v_1, v_2, \ldots, v_m\}$and $B_2 = \{w_1, w_2, \ldots, w_n\}$ be bases for
be bases for vector spaces V and W respectively.Let $\bar{0}_V, \bar{0}_W$ be zero vectors in V and W respectively. Then $B = (B_1 \times \{\bar{0}_W\}) \cup (\{\bar{0}_V\} \times B_2)$ is basis for product vector space $V \times W$. Moreover, if for $v \in V$, $(v)_{B_1} = (a_1, a_2, a_3, \ldots a_m)$and for $w \in W$, $(w)_{B_2} = (b_1, b_2, b_3, \ldots b_n)$are coordinate vectors of $v \in V$ and$w \in W$ with respect to bases B_1 and B_2respectively then coordinate vector of $(v,w) \in V \times W$with respect to basis B is $(v,w)_B = (a_1, a_2, a_3, \ldots a_m, b_1, b_2, b_3, \ldots b_n) = ((v)_{B_1}, (w)_{B_2})$
.

Proof: By theorem 2, B is a basis for product vector space $V \times W$.
Now to show $(v,w)_B = (a_1, a_2, a_3, \ldots a_m, b_1, b_2, b_3, \ldots b_n)$.
For this suppose
$$(v,w) = c_1(v_1, \bar{0}_W) + c_2(v_2, \bar{0}_W) + \cdots + c_m(v_m, \bar{0}_W) + d_1(\bar{0}_V, w_1) + d_2(\bar{0}_V, w_2) + \ldots + d_n(\bar{0}_V, w_n)$$
.
$$\Rightarrow v = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m \text{ , } w = d_1 w_1 + d_2 w_2 + \cdots + d_n w_n \qquad (1)$$
Now,
$(v)_{B_1} = (a_1, a_2, a_3, \ldots a_m)$ and $(w)_{B_2} = (b_1, b_2, b_3, \ldots b_n)$
$$\Rightarrow v = a_1 v_1 + a_2 v_2 + \cdots + a_m v_m \text{ , } w = b_1 w_1 + b_2 w_2 + \cdots + b_n w_n \qquad (2)$$
From (1) and (2), $c_i = a_i$ and $d_j = b_j$, for each i and j.
$\therefore (v,w)_B = (a_1, a_2, a_3, \ldots a_m, b_1, b_2, b_3, \ldots b_n) = ((v)_{B_1}, (w)_{B_2})$.
Thus result is proved.

Theorem 5: Let $B_1 = \{v_1, v_2, \ldots, v_m\}$and $B_2 = \{w_1, w_2, \ldots, w_n\}$ be bases for
be bases for vector spaces V and W respectively.Let $\bar{0}_V, \bar{0}_W$ be zero vectors in V and W respectively. Then for $u \in V$
$B = (B_1 \times \{\bar{0}_W\}) \cup (\{u\} \times B_2)$ is basis for product vector space $V \times W$. Moreover, if for $v \in V$, $(v)_{B_1} = (a_1, a_2, \ldots a_m)$and for $w \in W$,

$(w)_{B_2} = (b_1, b_2, b_3, \dots b_n)$ are coordinate vectors of $v \in V$ and $w \in W$ with respect to bases B_1 and B_2 respectively then coordinate vector of $(v,w) \in V \times W$ with respect to basis B is

$$(v,w)_B = \left((v - (b_1 + b_2 + b_3 + \dots + b_n)u)_{B_1}, (w)_{B_2}\right)$$

.

Proof: By theorem 3, B is a basis for product vector space $V \times W$.

Now to show

$$(v,w)_B = \left((v - (b_1 + b_2 + b_3 + \dots + b_n)u)_{B_1}, (w)_{B_2}\right)$$

.

For this suppose

$$(v,w) = c_1(v_1, \bar{0}_W) + c_2(v_2, \bar{0}_W) + \dots + c_m(v_m, \bar{0}_W) + d_1(u, w_1) + \dots + d_n(u, w_n)$$

$$\Rightarrow v = c_1v_1 + c_2v_2 + \dots + c_mv_m + (d_1 + d_2 + \dots + d_n)u,$$

$$and \quad w = d_1w_1 + d_2w_2 + \dots + d_nw_n$$

Now,

$$(w)_{B_2} = (b_1, b_2, b_3, \dots b_n) \Rightarrow w = b_1w_1 + b_2w_2 + \dots + b_nw_n$$

From this, $d_j = b_j$ for each j.

Substituting these values in expression for v we get,

$$v = c_1v_1 + c_2v_2 + \dots + c_mv_m + (b_1 + b_2 + \dots + b_n)u$$

$$\therefore v - (b_1 + b_2 + \dots + b_n)u = c_1v_1 + c_2v_2 + \dots + c_mv_m$$

Since B_1 is a basis for vector space, there exist $c_1, c_2, \dots, c_m$ in field F satisfying above equation.

$$\therefore (v - (b_1 + b_2 + \dots + b_n)u)_{B_1} = (c_1, c_2, \dots c_m)$$

$$\therefore (v,w)_B = (c_1, c_2, c_3, \dots c_m, b_1, b_2, b_3, \dots b_n) = \left((v - (b_1 + b_2 + \dots + b_n)u)_{B_1}, (w)_{B_2}\right).$$

Thus result is proved.

Theorem 6: Let $B_1 = \{v_1, v_2, \dots, v_m\}$ and $B_2 = \{w_1, w_2, \dots, w_n\}$ be bases for

be bases for vector spaces V and W respectively.Let $\bar{0}_V, \bar{0}_W$ be zero vectors in V and W respectively. Then for $u \in W$

$B = (B_1 \times \{u\}) \cup (\{\bar{0}_V\} \times B_2)$ is basis for product vector space $V \times W$. Moreover, if for $v \in V$, $(v)_{B_1} = (a_1, a_2, \dots a_m)$ and for $w \in W$, $(w)_{B_2} = (b_1, b_2, b_3, \dots b_n)$ are coordinate vectors of $v \in V$ and $w \in W$ with respect to bases B_1 and B_2 respectively then coordinate vector of $(v,w) \in V \times W$ with respect to basis B is

$$(v,w)_B = \left((v)_{B_1}, (w - (a_1 + a_2 + a_3 + \dots + a_m)u)_{B_2}\right)$$

Proof: Similar to the proof oh theorem 5.

2. Examples:

Firstly we study uncommonexamples of vector spaces to study basis and coordinate vectors in product vector spaces.

i) Let $V = \{x \in R \,/\, x > 0\}$. Define addition $(+)$ and scalar multiplication $(.)$ on V as follows: $x + y = xy$ and $.x = x^k$; $\forall x, y \in V$, $k \in R$. Then V is a one dimensional real vector space.

<u>**Solution:**</u> Step I: To show that V is a real vector space.

Let $x, y, z \in V$ and $k, l \in R$, then

C1: Additive Closure axiom: $x + y = xy \in V$.

C2: Multiplicative Closure axiom : $k.x = x^k \in V$.

A1: Additive associative axiom:

$(x + y) + z = (xy) + z = (xy)z = xyz$ (1)

$x + (y + z) = x + (yz) = x(yz) = xyz$ (2)

From (1) and (2) vector addition is associative operation on V. A2: Additive commutative axiom: $x + y = (xy) = (yx) = y + x$

A3: Existence of additive identity (zero) vector: Suppose for $x \in V$ there is $\bar{0}$ such that

$x + \bar{0} = x \Rightarrow x\bar{0} = x \Rightarrow \bar{0} = 1 \in V$ **is the zero vector.**

A4: Existence of negative vector: Suppose for $x \in V$ there is a vector y such that

$x + y = 1 \Rightarrow xy = 1 \Rightarrow y = \frac{1}{x} \subset V$ **is the negative vector of** $x \in V$.

M1:

$$k.(x + y) = k.(xy) = (xy)^k = x^k y^k = x^k + y^k = k.x + k.y$$

M2:

$$(k + l).x = x^{k+l} = x^k x^l = x^k + x^k = k.x + l.x$$

M3: $k.(l.x) = k.x^l = (x^l)^k = x^{kl} = (kl).x$

M4: $1.x = x^1 = x$

Thus V satisfy all axioms of vector space over the field R.

Therefore, **V is a real vector space.**

Step II: To show $dim.V = 1$.

For this consider $B = \{c\}$, $c \in V - \{1\}$.Then for $k \in R$ we have,

$k.c = 1 = \bar{0} \Rightarrow c^k = 1 = c^0 \Rightarrow k = 0$.

This shows B is linearly independent set in V. (3)

Again for $x \in V$ suppose there is a real number k such that

$$k.c = x \Rightarrow c^k = x$$

$$\Rightarrow k\log c = \log x$$

$\Rightarrow k = \frac{\log x}{\log c} \in R$ is such that $k.c = x$, where $x \in V$ is arbitrary.

$\therefore L(B) = V$ = linear span of B.(4)

From (3) and (4) B is a basis for V and it contain only one vector.

$$\mathbf{dim.V = 1.}$$

ii) Let $V = \{x \in R \,/\, x < 0\}$. Define addition $(+)$ and scalar multiplication $(.)$ on V as follows: $x + y = -(xy)$ and $.x = -(-x)^k$; $\forall x, y \in V$, $k \in R$. Then V is a one dimensional real vector space.

Solution: Step I: To show that V is a real vector space.
Let $x, y, z \in V$and $k, l \in R$, then
C1: Addititivity Closure axiom:$x + y = -(xy) \in V$.
C2: Multiplicative Closure axiom: $k.x = -(-x)^k \in V$.
A1: Additive associative axiom:
$(x + y) + z = [-(xy)] + z = -[-(xy)]z = xyz$ (1)
$x + (y + z) = x + [-(yz)] = -x[-(yz)] = xyz$ (2)
From (1) and (2) vector addition is associative operation on V.A2: Additive commutative axiom: $x + y = -(xy) = -(yx) = y + x$
A3: Existence of additive identity (zero) vector: Suppose for $x \in V$ there is $\bar{0}$ such that
$x + \bar{0} = x \Rightarrow -x\bar{0} = x \Rightarrow \bar{0} = -1 \in V$**isthe zero vector.**
A4: Existence of negative vector: Suppose for $x \in V$ there is a vector y such that
$x + y = -1 \Rightarrow -xy = -1 \Rightarrow y = \frac{1}{x} \in V$**is the negative vector of $x \in V$.**
M1:
$k.(x + y) = k.(-xy) = -(-(-xy))^k = -(xy)^k = -[(-x)(-y)]^k = -((-x)^k(-y)^k) = -(-(-x)^k.-(-y)^k) = -(k.x\, k.y) = k.x + k.y$

M2:
$(k + l).x = -(-x)^{k+l} = -[(-x)^k(-x)^l] = -(-(-x)^k)(-(-x)^l) = -((k.x)(k.y)) = k.x + k.y$

M3:
$k.(l.x) = k.[-(-x)^l] = -\left[-[-(-x)^l]\right]^k = -(-x)^{kl} = (kl).x$

M4: $1.x = -(-x)^1 = x$
Thus V satisfy all axioms of vector space over the field R.
Therefore, **V is a real vector space.**
Step II: To show $dim.V = 1$.For this consider $B = \{c\}$, $c \in V - \{-1\}$.Then for $k \in R$ we have,
$k.c = -1 = \bar{0} \Rightarrow -(-c)^k = -1 \Rightarrow (-c)^k = 1 = (-c)^0 \Rightarrow k = 0$.
This shows B is linearly independent set in V. (3) Again for $x \in V$ suppose there is a real number k such that
$k.c = x \Rightarrow -(-c)^k = x \Rightarrow (-c)^k = -x > 0$
$\Rightarrow k log(-c) = log(-x)$
$\Rightarrow k = \frac{log(-x)}{log(-c)} \in R$is such that $k.c = x$, where $x \in V$ is arbitrary.
$\therefore L(B) = V$= linear span of B(4)
From (3) and (4) B is a basis for V and it contain only one vector.
$dim.V = 1.$

iii) Let
$V - R^2 \times R^+ - \{(x,y,z)/\, x, y \epsilon R \,\&\, z > 0\}$
.
For $x = (x_1, y_1, z_1), y = (x_2, y_2, z_2) \epsilon V$ vector addition (+) and scalar multiplication (.) are defined as
$x + y = (x_1 + x_2, y_1 + y_2, z_1 z_2)$and
$k.x = (kx_1, ky_1, z_1{}^k), k \epsilon R$.
Show that a) V is a real vector space.
b) Show that the set $B = \{(1,0,1), (1,1,1), (0,0,5)\}$ is a basis for V.
c) Find the coordinate vector $(-5, 10, 6)_B$.

Solution: a) We know that $U = R^2$ is real vector space with usual addition of vectors and usual scalar multiplication operations whose **zero vector is $(0,0)$**. The set $W = R^+$ is also real vector space with operations $z_1 + z_2 = z_1 z_2$ and $k.z = z^k$ for $z_1, z_2, z \epsilon R^+$ and $k \epsilon R$.**The zero vector of R^+ is 1.**

$\therefore V$is a is a real vector space with respect to defined operations.
b) Now $B_1 = \{(1,0), (1,1)\}$and $B_2 = \{(5)\}$are bases for vector spaces R^2andR^+.
$\therefore B = (B_1 \times \{\bar{0}_W\}) \cup (\{\bar{0}_U\} \times B_2) = \{(1,0,1), (1,1,1), (0,0,5)\}$
is basis for V.
c) Now
$(-5,10) = -15(1,0) + 10(1,1) \Rightarrow (-5,10)_{B_1} = (-15,10)$

For
$6 \epsilon W, 6 = c.5 \Rightarrow 6 = 5^c \Rightarrow c = \frac{log6}{log5} \Rightarrow (6)_{B_2} = \left(\frac{log6}{log5}\right)$
.
$\therefore (-5, 10, 6)_B = ((-5,10)_{B_1}, (6)_{B_2}) = \left(-15, 10, \frac{log6}{log5}\right)$
.Let
$V = R^2 \times R^- = \{(x, y, z)/x, y \epsilon R \,\&\, z < 0\}$
.

For $x = (x_1, y_1, z_1), y = (x_2, y_2, z_2) \epsilon V$ vector addition (+) and scalar multiplication (.) are defined as

$x + y = (x_1 + x_2, y_1 + y_2, -z_1 z_2)$ and

$k.x = (kx_1, ky_1, -(-z_1{}^k), k\epsilon R$.

Show that a) V is a real vector space.

b) Show that the set

$B = \{(1,0,-1),(1,1,-1),(0,0,-5)\}$ is a basis for V.

c) Find the coordinate vector $(-5,10,-6)_B$.

Solution: a) We know that $U = R^2$ is real vector space with usual addition of vectors and usual scalar multiplication operations whose **zero vector is** $(0,0)$. The set $W = R^+$ is also real vector space with operations $z_1 + z_2 = z_1 z_2$ and $k.z = z^k$ for $z_1, z_2, z \epsilon R^+$ and $k \epsilon R$. **The zero vector of R^- is -1.**

$\therefore V$ is a is a real vector space with respect to defined operations.

b) Now $B_1 = \{(1,0),(1,1)\}$ and $B_2 = \{(-5)\}$ are bases for vector spaces R^2 and R^-.

$\therefore B = (B_1 \times \{\bar{0}_W\}) \cup (\{\bar{0}_U\} \times B_2) = \{(1,0,-1),(1,1,-1),(0,0,-5)\}$

is basis for V.

c)Now

$(-5,10) = -15(1,0) + 10(1,1) \Rightarrow (-5,10)_{B_1} = (-15,10)$

For

$-6 \epsilon W, -6 = c.-5 \Rightarrow 6 = -(-(-5)^c \Rightarrow c = \frac{log6}{log5} \Rightarrow (-6)_{B_2} = \left(\frac{log6}{log5}\right)$

.

$\therefore (-5,10,-6)_B = \left((-5,10)_{B_1}, (6)_{B_2}\right) = \left(-15,10,\frac{log6}{log5}\right)$

.

iv) Let $V = R^+ \times R^- = \{(x,y)/\ x > 0, y < 0\}$.

For $x = (x_1, x_2)$,

$y = (y_1, y_2) \epsilon V$ vector addition and scalar multiplication operations are defined as $x + y = x_1 x_2 \ and\ k.x = (x_1{}^k, -(-x_2)^k)$. Show that

a) $B_1 = \{(6,-1),(5,-10)\}\ and\ B_2 = \{(5,-10),(1,-50)\}$

are bases for real vector space V.

Solution: Here R^+ is a real vector space with operations $x_1 + x_2 = x_1 x_2$,

$k.x - x^k$ for $x_1, x_2 \epsilon R^+\ and\ k \epsilon R$ whose zero vector is 1.

Similarly, R^- is a real vector space with operations $x_1 + x_2 = -x_1 x_2$,

$k.x = -(-x)^k$ for $x_1, x_2 \epsilon R^+\ and\ k \epsilon R$ whose zero vector is -1.

a) Method1: Since any nonzero vector is basis vector for both one dimensional vector spaces. Therefore, $\{6\}$ and $\{-10\}$ are basis for vector spaces R^+ and R^- respectively.

$\therefore \{6\} \times \{-1\} \cup \{5\} \times \{-10\} = \{(6,-1),(5,-10)\} = B_1$

and

$\{5\} \times \{-10\} \cup \{1\} \times \{-50\} = \{(5,-10),(1,-50)\} = B_2$

are bases for product vector space $V - R^+ \times R^-$. **(By theorem 3)**

Method 2: By using definition of basis we show that $B_1\ and\ B_2$ are linearly independent and span vector space V.

For linear independence of B_1 , consider

$a(6,-1) + b(5,-10) = (1,-1) = zero\ vector\ in\ V.$

$\therefore \left(6^a 5^b, -[-(-1)]^a[-(-10)]^b\right) = (1,-1).$

$\therefore 6^a 5^b = 1, -(10^b) = -1.$

$\therefore b = 0, a = 0$.

Therefore B_1 is linearly independent in V. (1)

For linear span consider

$a(6,-1) + b(5,-10) = (x,y) = arbitrary\ vector\ in\ V.$

$\therefore \left(6^a 5^b, -[-(-1)]^a[-(-10)]^b\right) = (x,y).$

$\therefore 6^a 5^b = x, -(10^b) = y$.

$\therefore b = \frac{\log(-y)}{log\ 10} = c\ in\ R\ as\ y\ is\ negative.$

Then $6^a = x\ 5^{-b} \Rightarrow a = \frac{\log(5^{-b}x)}{log\ 6} \epsilon R.$

From this any vector in V is expressed as linear combination of basis vectors.

$\therefore L(B_1) = V$.(2)

From (1) and (2) B_1 is basis for V.

Similarly, B_2 is basis for V.

v) Let $V = R^3 \times R^+ \times R^-$.

For $x = (x_1, x_2, x_3, x_4, x_5)$, $y = (y_1, y_2, y_3, y_4, y_5) \epsilon V$ and $k \epsilon R$,

$x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 y_4, x_5 y_5)$ and

$.x - \left(kx_1, kx_2, kx_3, x_4^k, -(-x_5)^k\right)$. Then show that V is a real vector space.

$B =$
$\{(1,1,1,1,-1),(1,1,0,1,-1),(1,0,0,1,-1)),(0,0,0,5,-1),(0,0,0,1,-7)\}$ isbasis for V.Also, Find the coordinator vector of $v = (1,2,3,4,-5)$ relative to the basis B.

Solution: Here, $R^+$is a real vector space w.r.t. operations
$(x_1,x_2,x_3)+(y_1,y_2,y_3) - (x_1+y_1,x_2+y_2,x_3+y_3)$
and
$k.(x_1,x_2,x_3) = (kx_1,kx_2,kx_3)$.

Similarly $R^+$and R^- are real vector spaces w.r.t. operations defined in examples i) and ii).

Therefore $V = R^3 \times R^+ \times R^-$ is a real vector space w.r.t. operations defined here as it is product vector space.

Now, $B_1 = \{(1,1,1),(1,1,0),(1,0,0)\}$, $B_2 = \{5\}, B_3 = \{-7\}$ are bases and $(0,0,0), 1, -1$ are zero vectors for real vector spacesR^3, R^+, R^-respectively.

$\therefore B = (B_1 \times \{1\} \times \{-1\}) \cup ((0,0,0) \times B_2 \times \{-1\}) \cup ((0,0,0) \times \{1\} \cup B_3)$

is a basis for vector space V.

Now in R^3 , $(1,2,3) = 3(1,1,1) - 1(1,1,0) - 1(1,0,0)$
$\therefore (1,2,3)_{B_1} = (3,-1,-1)$.

Similarly in $R^+, 4 = k.5 = 5^k \Rightarrow k = \frac{log4}{log5} \Rightarrow (4)_{B_2} = \left(\frac{log4}{log5}\right)$.

And in $R^-, -5 - k.(-7) - -7^k \rightarrow k - \frac{log5}{log7} \rightarrow (-5)_{B_3} - \left(\frac{log5}{log7}\right)$

From this $(1,2,3,4,-5)_B = ((1,2,3)_{B_1},(4)_{B_2},(-5)_{B_3}) = \left(3,-1,-1,\frac{log4}{log5},\frac{log5}{log7}\right)$

.

CONCLUSION

In this paper we studied basis for product vector space and its application to find the coordinate vectors in terms of coordinate vectors of vectors lying in vector spaces whose vector product is defined.

ACKNOWLEDGMENT

I am very much thankful to **Dr.P.G.Jadhav**, Department of Mathematics, ShriBalasahebJadhavArts, commerce and Science College, Ale,Associate Professor **J.P.Jadhav**,Head,Department of Mathematics, H.P.T. Arts and R.Y.K.Science College, Nasik and Assistant professor **Miss R.N. Tilak**, Department of Mathematics, H.P.T. Arts and R.Y.K. Science College, Nasik, for their valuable suggestions and continuous guidance to complete this paper.

REFERENCES

1. **K.Hoffman and R. Kunze**, *Linear Algebra(second edition)*, Prentice Hall of India,New Delhi, (1998).
2. **S. LangS.**,Introduction to Linear *Algebra,Second Ed. Springer-Verlag, New Yark,Wesley*,(1967)
3. **Kenneth Kuttler,** *Linear Algebra, Theory and Applications, E-book (2013)*
4. **KantiBhushanDatta,** *Matrix and Linear Algebra aided with MATLAB, PHI Learning Pvt.Ltd, New Delhi (2009).*
5. **Howard Anton, Chris Rorres.**, *Elementary Linear Algebra, John Wiley and Sons, Inc.*
6. **Herstein I.N.**, *Topics in Algebra*, John Wiley, (1975).
7. **J. B. Fraleigh,** *A first course in Abstract Algebra, Thirdedition, Narosa, New Delhi,(1990)*
8. **Joseph**, A. **Gallian,** *Compulsory Abstract Algebra, fourth edition, Narosa Publication House*
9. **Divate B.B.,** Study of Product of Vector spaces and its applications, *International Journal of scientific and innovative mathematics,Vol.3,special issue-2(July 2015)*