MNK Publication

International Journal of Latest Research in Science and Technology **ISSN (Online):2278-5299**
Volume 4, Issue 6: Page No.91-93, November-December 2015 (special Issue Paper)
http://www.mnkjournals.com/ijlrst.htm
Special Issue on International conference on Methematics Sciences-2015(MSC-2015)
Confernece Held at Sadguru Gadage Maharaj College, Karad ,Maharastra,(India)

SIMULTANEOUS APPROXIMATION BY SUMMATION – INTEGRAL TYPE OPERATORS

SANTOSH V. BABAR

Department of Mathematics,Dr. Ghali College,Gadhinglaj, Dist. Kolhapur,Maharashtra - 416502.santosh.babar425@gmail.com

Abstract- ***In this paper, we consider a general sequence of summation-integral type operators. The aim of the paper is to study some direct result of summation-integral type operators for functions of bounded variation.***

I. INTRODUCTION

In the year 2003 Srivastava and Gupta[4] investigated as well as estimated the rate of convergence of the general sequence of operators $G_{n,c}$ by means of the decomposition technique for functions of bounded variation. Also Ispir and Yukesel[2] introduced the Bezier variant of these operators and estimated the rate of convergence for function of bounded variations.

Srivastava and Gupta defined a summation-integral type operators $G_{n,c}$ as follows,

$$G_{n,c}(t,x)=n\sum_{k=1}^{\infty}P_{n,k}(x,c)\int_{0}^{1}P_{n+c,k-1}(t,c)\,f(t)\,dt+P_{n,0}(x,c)\,f(0)$$
..... (1)

where $P_{n,k}(n,c)=\dfrac{(-x)^k}{k!}\phi_{n,c}^{k}(x)$

and $\phi_{n,c}(x)=\begin{cases} e^{-nx} & ,c=0 \\ (1+cx)^{-n/c} & ,c=1,2,3,..... \end{cases}$

Here $\{\phi_{n,c}(x)\}_{n=1}^{\infty}$ is a sequence of functions defined on the closed interval

[0, b], $b > 0$ which satisfy the following properties for every $n \in \mathbb{N}$ and $k \in N_0 = N \cup \{0\}$.

(i) $\phi_{n,c} \in c^{\infty}[a,b]$ $(b > a \geqq 0)$

(ii) $\phi_{n,c}(0) = 1$

(iii) $\phi_{n,c}(x)$ is completely monotone so that $(-1)^k \phi_{n,c}^{k}(x) \geqq 0$ $(0 \leqq x \leqq b)$

(iv) There exist an integer c such that,

$$\phi_{n,c}^{k+1}(x) = -n\,\phi_{n+c,c}^{(k)}(x)$$
$$(n > \max\{0,-c\};\, x \in [0,b])$$

2) CONSTRUCTION OF THE OPERATORS

In this section, the operators $G_{n,c}(f,x)$ defined by (1) can also be considered when $c=-1$, we have

$$G_{n,-1}(f,x)=n\sum_{k=1}^{n}P_{n,k}(x;-1)\int_{0}^{1}P_{n-1,k-1}(t;-1)\,f(t)\,dt+(1-x)^n f(0)$$
..... (2)

$$(f \in H_{\alpha}(0,1);\, 0 \leqq x \leqq 1)$$

where, $P_{n,k}(x;-1)=\binom{n}{k}x^k(1-x)^{n-k}$
..... (3)

On the other hand, the general operators defined by (2) can alternatively be written in the form,

$$G_{n,-1}(f;x)=\int_{0}^{1}k_n(x,t;-1)\,f(t)\,dt$$
$$(f \in H_{\alpha}(0,1);\, 0 \leqq x \leqq 1) \quad .... (4)$$

where,

$$k_n(x,t;-1)=n\sum_{k=1}^{n}P_{n,k}(x,-1)P_{n-1,k-1}(t;-1)+(1-x)^n(1-t)^n\delta(t)$$
..... (5)

In the present paper, we estimated direct results of the operator $G_{n,-1}$ by means of the decomposition technique for functions of bounded variation using auxiliary function $g_x(t)$ which is defined by

$$g_x(t)=\begin{cases} f(t)-f(x-) & (0 \leqq t < x) \\ 0 & (t = x) \\ f(t)-f(x+) & (x < t < \infty) \end{cases} \quad ..... (6)$$

3) AUXILLIARY RESULTS

In order to prove our main result we require following lemmas.

Lemma -1[3]. For all $x \in (0,\infty)$ and $k \in \mathbb{N}$,

$$P_{n,k}(x,-1) \leqq \frac{1}{\sqrt{2e}} \frac{1}{\sqrt{nx(1-x)}} \text{ ..... (6)}$$

Lemma-2[4] : Let

$$\mu_{n,m}(x;-1) = n\sum_{k=1}^{\infty} P_{n,k}(x;-1)\int_0^1 P_{n-1,k-1}(t;-1)(t-x)^m dt + (-x)^m P_{n,0}(x;-1) \text{ .... (7)}$$

Then, $\mu_{n,0}(x;-1)=1$, $\mu_{n,1}(x;-1)=\frac{-x}{n+1}$

and $\mu_{n,2}(x;-1)=\frac{x(1-x)(2n+1)+(1-3x)x}{(n+1)(n+2)}$ .... (8)

In particular, given any number $x > 0$ lemma 2 yields the inequality,

$$\mu_{n,2}(x;-1) \leqq \frac{\lambda x(1-x)}{n} \ (\lambda > 2) \text{..... (9)}$$

Lemma -3[4]. Let $x \in (0,1)$ and $k_n(x,t;-1)$ be defined by (5). Then for $\lambda > 2$ and for sufficiently large n,

$$B_n(x,y) = \int_0^y k_n(x,t;-1)\,dt \leqq \frac{\lambda x(1-x)}{n(x-y)^2}$$

$(0 \leqq y < x)$ ..... (10)

and $$1 - B_n(x,z) = \int_z^1 k_n(x,t;-1)\,dt \leqq \frac{\lambda x(1-x)}{n(z-x)^2}$$

$(x < z \leq 1)$..... (11)

Proof : Since $0 \leqq y < x$, for $t \in [0,y]$ we have,

$$\frac{x-t}{x-y} \geq 1$$

from definition (4) we find that,

$$\int_0^y k_n(x,t;-1)\,dt \leqq \int_0^y \left(\frac{x-t}{x-y}\right)^2 k(x,t;-1)\,dt \leqq (x-y)^{-2}\mu_{n,2}(x;-1) \leqq \frac{\lambda x(1-x)}{n(x-y)^2}$$

$(0 \leqq y < x)$

The proof of inequality (11) is similar.

4) MAIN RESULTS

In this section, we prove the following result.

Theorem : Let f be a function of bounded variation on every finite sub-interval of the closed interval [0, 1]. Suppose also that the one-sided limits $f(x-)$ and $f(x+)$ exist for some fixed point $x \in (0,1)$. Then, for $\lambda > 2$ and sufficiently large n,

$$\left|G_{n,-1}(f:x) - \frac{1}{2}\{f(x+)+f(x-)\}\right| \leqq \frac{1}{\sqrt{4enx(1-x)}}\left|f(x+)+f(x-)\right| + \frac{[2\lambda(1-x)+x]}{nx}$$

$$\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(gx) \text{ ..... (12)}$$

where, $$g_x(t) = \begin{cases} f(t) - f(x-) & (0 \leqq t < x) \\ 0 & t = x \\ f(t) - f(x+) & (x < t < 1) \end{cases} \text{ .... (13)}$$

and $V_a^b(g_x)$ denotes total variation of g_x on [*a*, *b*].

Proof : In our proof of theorem first.

$$\left|G_{n,-1}(f,x) - \frac{1}{2}\{f(x+)-f(x-)\}\right| \leqq \left|G_{n,-1}(g_x,x)\right| + \frac{1}{2}\left|f(x+)-f(x-)\right| \cdot \left|G_{n,-1}(\text{sign}(t-x),x)\right| \text{(14)}$$

We need estimates for $G_{n,-1}(g_x,x)$ and $G_{n,-1}(\text{sign}(t-x),x)$.

To estimates $G_{n,-1}(\text{sign}(t-x),x)$, we first observe that,

$$G_{n,-1}(\text{sign}(t-x),x) = \int_0^1 k_n(x,t;-1)\,\text{sign}(t-x)\,dt$$

$$= \int_x^1 k_n(x,t;-1)\,dt - \int_0^x k_n(x,t;-1)\,dt$$

$$= 2\int_x^1 k_n(x,t;-1)\,dt - 1 \text{ .... (15)}$$

$$\text{since} \int_0^1 k_n(x,t;-1)\,dt = 1$$

.... (16)

But, $$n\int_x^1 P_{n-1,k}(t;-1)\,dt = \sum_{j=0}^{k} P_{n,j}(x;-1)$$

Thus we obtain,

$$\int_x^1 k_n(x,t;-1)\,dt = n\sum_{k=1}^{n} P_{n,k}(x;-1)\int_x^1 P_{n-1,k-1}(t;-1)\,dt + (1-x)^n \int_x^1 \delta(t)\,dt$$

$$= n\sum_{k=1}^{n} P_{n,k}(x;-1)\sum_{j=0}^{k-1} P_{n,j}(x;-1)$$

Here, $\delta(t) = 0$ for $t \geqq x > 0$.

Since

$$I = \left[P_{n,0}(x,-1)+P_{n,1}(x,-1)+P_{n,2}(x,-1)+\ldots\right]\left[P_{n,0}(x,-1)+P_{n,1}(x,-1)+\ldots\right]$$

It is easily verified that,

$$G_{n,-1}(\text{sign}(t-x),x) = 2\int_x^1 k_n(x,t;-1)\,dt - 1 = P_{n,0}^2(x,-1) + P_{n,1}^2(x,-1) + P_{n,2}^2(x,-1) + \ldots$$

Hence by lemma (1) we have,

$$\left|G_{n,-1}(\text{sign}(t-x),x)\right| \leqq \frac{1}{2\sqrt{e}} \frac{1}{\sqrt{nx(1-x)}} \sum_{k=0}^{n} P_{n,k}(x,-1) = \frac{1}{\sqrt{4enx(1-x)}} \text{ .... (17)}$$

Next to estimate of $G_{n,-1}(g_x, x)$ we have,

$$G_{n,-1}(g_x, x) = \int_0^1 g_x(t) k_n(x,t;-1)\,dt$$

$$= \left[\int_0^{x-\frac{x}{\sqrt{n}}} + \int_{x-\frac{x}{\sqrt{n}}}^{x-\frac{1-x}{\sqrt{n}}} + \int_{x+\frac{1-x}{\sqrt{n}}}^{1}\right] k_n(x,t;-1)\, g_x(t)\,dt = E_1 + E_2 + E_3 \quad \text{..... (18)}$$

For $t \in \left[x - \frac{x}{\sqrt{n}} - x + \frac{(1-x)}{\sqrt{n}}\right]$ we have,

$$|g_x(t)| \leqq V_{x-\frac{x}{\sqrt{n}}}^{x+\frac{(1-x)}{\sqrt{n}}}(g_x) \leqq \frac{1}{n}\sum_{k=1}^{n} V_{x-\frac{x}{\sqrt{n}}}^{x+\frac{(1-x)}{\sqrt{n}}}(g_x)$$

and so

$$|E_2| \leqq V_{x-\frac{x}{\sqrt{n}}}^{x+\frac{(1-x)}{\sqrt{n}}}(g_x) \leqq \frac{1}{n}\sum_{k=1}^{n} V_{x-\frac{x}{\sqrt{n}}}^{x+\frac{(1-x)}{\sqrt{n}}}(g_x)$$

.... (19)

In order to estimate E_1, we set $y = x - \frac{x}{\sqrt{n}}$ and integrate by parts; we thus obtain

$$E_1 = \int_0^y g_x(t)\,dt\left(B_n(x,t)\right) = g_x(y) B_n(x,y) - \int_0^y B_n(x,t)\,dt\left(g_x(t)\right)$$

Since $|g_x(y)| \leqq V_y^x(g_x)$ conclude that

$$|E_1| \leqq V_y^x(g_x) B_n(x,y) + \int_0^y B_n(x,t)\,dt\left(-V_t^x(g_x)\right)$$

For $y = x - \frac{x}{\sqrt{n}} \leqq x$ by using lemma (3), we get,

$$|E_1| \leqq \frac{\lambda x(1-x)}{n(x-y)^2} V_y^x(g_x) + \frac{\lambda x(1-x)}{n}\int_0^y \frac{1}{(x-t)^2}\,dt\left(-V_t^x(g_x)\right)$$

Integrating the last integral by parts, we get

$$|E_1| \leqq \frac{\lambda x(1-x)}{n}\left(x^{-2} V_0^x(g_x) + 2\int_0^y \frac{V_t^x(g_x)}{(x-t)^3}\,dt\right) \quad \text{.... (20)}$$

for which $y = x - \frac{x}{\sqrt{n}}$, yields

$$\int_0^{x-\frac{x}{\sqrt{n}}} V_t^x(g_x)(x-t)^{-3}\,dt = \sum_{k=1}^{n-1}\int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{(1-x)}{\sqrt{k}}} V_{x-t}^x(g_x)\, t^{-3}\,dt \leqq \frac{1}{2x^2}\sum_{k=1}^{n} V_{x-\frac{x}{\sqrt{k}}}^x(g_x)$$

Hence, $|E_1| \leqq \frac{\lambda(1-x)}{nx}\sum_{k=1}^{n} V_{x-\frac{x}{\sqrt{k}}}^x(g_x)$ ..... (21)

Using a similar method and lemma, we obtain,

$$|E_3| \leqq \frac{\lambda(1-x)}{nx}\sum_{k=1}^{n} V_x^{x+\frac{(1-x)}{\sqrt{k}}}(g_x) \quad \text{.... (22)}$$

From equations (19), (21) and (22), it follows that

$$|G_{n,-1}(g_x,x)| \leqq \frac{\lambda(1-x)}{nx}\sum_{k=1}^{n} V_{x-\frac{x}{\sqrt{k}}}^x(g_x) + \frac{1}{n}\sum_{k=1}^{n} V_{x-\frac{x}{\sqrt{k}}}^{x+\frac{(1-x)}{\sqrt{k}}}(g_x) + \frac{\lambda(1-x)}{nx}\sum_{k=1}^{n} V_{x-\frac{x}{\sqrt{k}}}^{x+\frac{(1-x)}{\sqrt{k}}}(g_x)$$

$$= \left[\frac{\lambda(1-x)}{nx} + \frac{\lambda(1-x)}{nx}\right]\sum_{k=1}^{n} V_{x-\frac{x}{\sqrt{k}}}^{x+\frac{(1-x)}{\sqrt{k}}}(g_x) + \frac{1}{n}\sum_{k=1}^{n} V_{x-\frac{x}{\sqrt{k}}}^{x+\frac{(1-x)}{\sqrt{k}}}(g_x)$$

$$= \left[\frac{2\lambda(1-x)}{nx} + \frac{1}{n}\right]\sum_{k=1}^{n} V_{x-\frac{x}{\sqrt{k}}}^{x+\frac{(1-x)}{\sqrt{k}}}(g_x)$$

$$|G_{n,-1}(g_x,x)| \leqq \frac{[2\lambda(1-x)+x]}{nx}\sum_{k=1}^{n} V_{x-\frac{x}{\sqrt{k}}}^{x+\frac{(1-x)}{\sqrt{k}}}(g_x)$$

..... (23)

Our theorem now follows from (19), (21) and (23). This completes the proof for the theorem.

REFERENCES :

1. D. K. Gupta and P. N. Agrawal, Convergence in simultaneous approximation for Srivastava-Gupta operators, Mathematical Sciences, 22(1) (2012), 1-8.
2. Nurhayat Ispir and Ismet Yuksel, on the benzier variant of Srivastava-Gupta Operators, App. Mathematics E-notes 5 (2005), 129-137.
3. X. M. Zeng, Bounds for Berstain Basis Functions and Meyer-Konig and Zeller Basis Functions, J. Math. Anal. Appl. 219, 364-376 (1998).
4. H. M. Srivastava and Vijay Gupta, A Certain Family of Summation - Integral Type Operators, Math. and Comp. Modeling, 37 (2003), 1307-1315.
5. X. M. Zeng and Wenzhong Chan on the Rate of Convergence of the Generalized. Durrmeyer Type Operators for Functions of Bounded Variation, J. Appr. Theory, 102 (2000), 1-12.