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NORMAL FUZZY BI-IDEALS OF A Γ-SEMI-NEARRING

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Abstract- ***In this paper, the concept of normal fuzzy bi-ideal of a Γ-semi-near ring is introduced and investigated some of its properties***

Key-words: Γ-semi-nearring , Fuzzy Γ-semi-nearrings, Fuzzy bi-ideals of a Γ-semi-nearring, Normal fuzzy bi-ideal of a Γ-semi-nearring

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I. INTRODUCTION

Saha N. K. and Sen S. K. et al [9-10] defined the concept of a Γ- semi-near-ring which is the generalization of Γ- near-ring, seminear ring and Γ- semiring. Pianskool S. et all[5] studied simple Γ- semi-near rings A paradigm shift occurred in the algebra due to Zadeh L. A. [11] in 1962 by introducing the concept of graded membership inplace of membership / nonmembership of ordinary set, namely fuzzy set. Further the notion of fuzzy group was introduced by Rosenfeld [8] and then fuzzy algebra is flourished by the contribution of many mathematicians.

In 1952, Good and Hughes introduced bi-ideals of semigroups. Chelvam T. T. and Ganesan N. [1] introduced bi-ideals of near rings. Pawar and Pandharpure [6] discussed ideals and bi-ideals in near rings. Chelvam T. T. and Meenakumari N. [2] studied bi-ideals of Γ -Near Rings. Fuzzy ideals and fuzzy bi-ideals of semigroups are studied by Kuroki N.[5]. Kim and Lee [4] studied intutionistic fuzzy bi-ideals of semigroup. Ezhilmaran and A. Dhandapani [3] have studied Characterization of Intutionistic Fuzzy Bi-ideals in Γ-semiring.

In this paper we study various properties of fuzzy bi-ideals of a $\Gamma - \text{seminearring}$.

§1.1. Preliminaries.

Throughout this section M denotes a Γ-semi-near ring unless otherwise specified. We begin with the following definition.

Definition 1.1.1. Let (M, +) be an additive semigroup and for α ∈ Γ, a nonempty set, (M, α) be a semigroup. Then M is called a right Γ-seminear ring if there exists a mapping M × Γ× M → M (denoted by (a, α, b) → aαb) satisfying the conditions:

i) (a+b) αc = aαc+bαc,

ii) aα(bβc) = (aαb)βc,

for all a, b, c ∈ M and α, β ∈ Γ.

Precisely speaking '**Γ-seminear ring**' to mean 'right Γ-seminear ring'.

Example 1.1.2. Let M = $\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ / a, b be nonnegative integers} = Γ, Then (M, +, α) is Γ- seminear ring under the matrix addition and matrix multiplication, α ∈ Γ. Define M × Γ× M → M (denoted by (a, α, b) → aαb) where aαb is matrix multiplication of a, α and b. Then M is a Γ-seminear ring.

Definition 1.1.3. Let M be a Γ-seminear ring. A nonempty subset I of M is a **sub-Γ-seminear ring** of M if I is also a Γ-seminear ring with the same operations of M.

Definition 1.1.4. A subset I of a Γ-seminear ring M is a **left (resp. right) ideal** of a Γ-seminear ring M if I is a subsemigroup of M and rαx ∈I (resp. xαr ∈I) for all x ∈ I and r ∈ M, α ∈ Γ. If I is both left as well as right ideal then we say that I is an **ideal of a Γ-seminear ring** M.

Example 1.1.5. Consider the example 1.1.2. of Γ-seminear ring (M, +, α), α ∈ Γ mentioned above. We have I = $\{\begin{bmatrix} 2a & 2b \\ 0 & 0 \end{bmatrix}$ / a, b be nonnegative integers} is an ideal of M.

Definition 1.1.6. A fuzzy subset μ_I of Γ-semi-nearring M is called a fuzzy Γ-semi-nearring of M if

(i) $\mu_I(x+y) \geq \min\{\mu_I(x), \mu_I(y)\}$,

(ii) $\mu_I(x\alpha y) \geq \min\{\mu_I(x), \mu_I(y)\}$,

for all α∈Γ and x, y ∈ M.

Definition 1.1.7. A sub-Γ-semi-nearring I of M is called a **bi-ideal** of Γ-semi-nearring M if IΓMΓI ⊆ I.

Definition 1.1.8. Let f: M → M' be mapping from Γ-semi-nearring M to Γ-semi-nearring M'. Then f is said to be homomorphic if f(x+y) = f(x) + f(y) and f(x α y) = f(x) α f(y).

§1.2. Fuzzy bi-ideals of a Γ-semi-nearring .

Definition 1.2.1: - A fuzzy sub-Γ-semi-nearring I of Γ-semi-nearring M is called a fuzzy bi-ideal of M if

(i) $\mu_I(x+y) \geq \min\{\mu_I(x), \mu_I(y)\}$,

(ii) $\mu_I(x\alpha y) \geq \min\{\mu_I(x), \mu_I(y)\}$,

(iii) $\mu_I(x\alpha u\beta y) \geq \min\{\mu_I(x), \mu_I(y)\}$,

for all α, β ∈ Γ and x, u, y ∈ M.

Example 1.2.2. From Example 1.1.2. and Example 1.1.5., I is a bi-ideal of M. Define μ: M→[0, 1] by letting μ(O) = 1, a zero matrix O ∈ M and μ(X)= 0.4 for all nonzero matrices X

$\in M$. Then μ , becomes fuzzy bi-ideal of M.Theroem 1.2.3. If

I is a bi-ideal of M then for any 0<t<1, there exists a fuzzy bi-ideal μ of M such that μ_t = I.

Proof. Let μ: M→[0, 1] be defined as

$$\mu(x) = \begin{cases} t, \text{ if } x \in I \\ 0, \text{ if } x \notin I \end{cases}$$

where t is a fixed number in (0, 1). Then μ_t = I.

Now suppose that I is a bi-ideal of M.

For all x, y ∈ I and β∈Γ such that xβy ∈ I, we have

$\mu(x+y) \geq t = \min\{\mu_I(x), \mu_I(y)\}$since $\mu(x) \geq t$

and $\mu(y) \geq t$ and

For all x, y ∈ I and β∈Γ such that xβy ∈ I, we have

$\mu(x\beta y) \geq t = \min\{\mu_I(x), \mu_I(y)\}$,

Also for all $\alpha, \beta \in \Gamma$ and x,u, y ∈ I such that xαuβy∈ I, we have
$\mu_A(x\alpha u\beta y) \geq t = \min\{\mu(x), \mu(y)\}$

Thus μ is a fuzzy bi-ideal of M.

Theorem 1.2.4. Let χ_I be the characteristic function of a subset I of M. Then χ_I is a bi-ideal of M if and only if I is a bi-ideal of M.Proof:If part.Let I be a bi-ideal of M.

Case 1: If x, y ∈I, then x + y ∈ I.

(a) If x, y ∈I, then x + y ∈ I. So χ_I (x) =1 = χ_I(y) and χ_I(x +y) = 1. Thus we get $\chi_I(x+y) = \min(\chi_I(x), \chi_I(y)) = 1$.
(b) If at least one of x and y does not belong to I, then $\chi_I(x+y) \geq \min(\chi_I(x), \chi_I(y)) = 0$.Since χ_I (x) =0 or χ_I(y) =0.
(c) If both x and y does not belong to I, then $\chi_I(x+y) \geq \min(\chi_I(x), \chi_I(y)) = 0$.Since χ_I (x) =0 or χ_I(y) =0.
Thus $\chi_I(x+y) \geq \min(\chi_I(x), \chi_I(y))$.

Case 2: If x, y ∈I and $\alpha \in \Gamma$, then x αy ∈ I.

(a) If x, y ∈I, then χ_I (x) =1 = χ_I(y) and χ_I(x αy) = 1. Thus we get $\chi_I(x\alpha y) = \min(\chi_I(x), \chi_I(y))$ =1.
(b) If at least one of x and y does not belong to I, then $\chi_I(x\alpha y) \geq \min(\chi_I(x), \chi_I(y)) = 0$.Since χ_I (x) =0 or χ_I(y) =0.
(c) If both x and y does not belong to I, then $\chi_I(x\alpha y) \geq \min(\chi_I(x), \chi_I(y)) = 0$.Since χ_I (x) =0 or χ_I(y) =0.

Therefore $\chi_I(x\alpha y) \geq \min(\chi_I(x), \chi_I(y))$ for all x, y ∈ M.

Case 3: If x, u, y ∈I and $\alpha, \beta \in \Gamma$, then xαuβy ∈ I.

(a) If x, y ∈I, χ_I (x) =1 = χ_I(y) and χ_I(x αuβy) = 1.

Thus we get $\chi_I(x\alpha u\beta y) = \min(\chi_I(x), \chi_I(y)) = 1$.

(b) If at least one of x and y does not belong to I, then $\chi_I(x\alpha u\beta y) \geq \min(\chi_I(x), \chi_I(y)) = 0$.Since χ_I (x) =0 or χ_I(y) =0.
(c) If both x and y does not belong to I, then $\chi_I(x\alpha u\beta y) \geq \min(\chi_I(x), \chi_I(y)) = 0$.Since χ_I (x) =0 or χ_I(y) =0.

Therefore $\chi_I(x\alpha u\beta y) \geq \min(\chi_I(x), \chi_I(y))$.

Thus χ_I is a fuzzy bi- ideal of M for all x, u, y ∈M and α, β ∈Γ.Only if part.Suppose χ_I be a fuzzy bi- ideal of M. Then by Theorem 1.2.3, I is a bi- ideal of M.

In the following theorem we discuss the properties of the image and preimage of fuzzy bi- ideal under a Γ-semi-near ring homomorphism.

Theorem 2.1.5. Let M and M′ be two Γ-near rings and f: M→M′ be a Γ-seminear ring homomorphism. Then

i) if μ is a fuzzy bi- ideal of M, then f(μ) is a fuzzy bi-ideal of M′,

ii) if γ is a fuzzy bi- ideal of M′, then $f^{-1}(\gamma)$ is a bi-ideal of M.Proof : i) Suppose μ is a fuzzy bi- ideal of M.

Let x, y ∈ M, u′, v′∈ M′ and α ∈ Γ.To prove that

$\{z/ z \in f^{-1}(u'+v')\} \supseteq \{x + y / x \in f^{-1}(u') \text{ and } y \in f^{-1}(v')\}$.

Let $x \in f^{-1}(u')$ and $y \in f^{-1}(v')$. Then f(x) = u′ and f(y) = v′.

This implies f(x) + f(y) = u′ + v′ ⟹f(x +y) = u′+ v′, since f is Γ-semi-near ring homomorphic. So $x + y \in f^{-1}(u' + v')$.

We have to prove that $\{z / z \in f^{-1}(u'\alpha v')\} \supseteq$

$\{x\alpha y / x \in f^{-1}(u') \text{ and } y \in f^{-1}(v')\}$.

$(f(\mu))(u'\alpha v') = \sup_{z\in f^{-1}(u'\alpha v')}\mu(z) \geq \sup\{\mu(x\alpha y)/f(x) = u', f(y) = v'\}$

$\geq \sup\{\min\{\mu(x), \mu(y)\}\}/f(x) = u', f(y) = v'\} \geq \min\{\sup\{\mu(x)/ f(x) = u'\}, \sup\{\mu(y)/ f(y) = v'\}\} = \min(\{f(\mu)(u')/ f(x) = u'\},\{f(\mu)(v')/ f(y) = v'\})$.

Therefore $(f(\mu))(u'\alpha v') \geq \min\{(f(\mu))(u'), (f(\mu))(v')\}$ and we have $\{z / z \in f^{-1}(u'\alpha v')\} \supseteq \{x\alpha y / x\in f^{-1}(u')$ and $y \in f^{-1}(v')\}$.

Also to prove $\{z / z \in f^{-1}(u'\alpha w'\beta v')\} \supseteq \{x\alpha z\beta y / x\in f^{-1}(u')$ and $y \in f^{-1}(v')\}$. Let $x\in f^{-1}(u')$ and $y\in f^{-1}v')$. So $f(x) = u'$, $f(z) = w'$ and $f(y) = v'$.

This implies $f(x)\alpha f(z)\beta f(y) = u'\alpha w'\beta v'$. $f(x\alpha z\beta y) = u'\alpha w'\beta v'$, since f is Γ-near ring homomorphic. So $x\alpha z\beta y \in f^{-1}(u'\alpha w'\beta v')$.

Thus $\{z / z \in f^{-1}(u'\alpha w'\beta v')\} \supseteq \{x\alpha z\beta y / x\in f^{-1}(u')$, $z\in f^{-1}(w')$ and $y \in f^{-1}(v')\}$

Hence $f(\mu)$ is a fuzzy bi- ideal of M′.

ii) Suppose γ is a fuzzy bi- ideal of M′.

$(f^{-1}\gamma))(x+y) = \gamma(f(x+y)) = \gamma(f(x)+f(y)) \geq \min\{\gamma(f(x)), \gamma(f(y))\} = \min\{(f^{-1}(\gamma))(x), (f^{-1}\gamma))(y)\}$.

Thus $(f^{-1}(\gamma))(x+y) \geq \min\{(f^{-1}(\gamma))(x), (f^{-1}(\gamma))(y)\}$ for all $x,y \in M$.

$(f^{-1}\gamma))(x\alpha y) = \gamma(f(x\alpha y)) = \gamma(f(x)\alpha f(y)) \geq \min\{\gamma(f(x)), \gamma(f(y))\} = \min\{(f^{-1}(\gamma))(x), (f^{-1}(\gamma))(y)\}$, for all $x, y \in M$ and $\alpha \in \Gamma$. Also, $(f^{-1}(\gamma))(x\alpha z\beta y)$

$= \gamma\{f(x\alpha z\beta y)\} = \gamma\{f(x)\alpha f(z)\beta f(y)\} \geq \min\{\gamma(f(x)), \gamma(f(y))\} = \min\{(f^{-1}(\gamma))(x), (f^{-1}\gamma))(y)\}$, for all $x, z, y \in M$ and $\alpha, \beta\in \Gamma$.

Hence $f^{-1}(\gamma)$ is a fuzzy bi- ideal of M.

§1.3. Normal Fuzzy bi-ideals of a Γ-semi-nearring .

Definition 1.3.1. A fuzzy bi-ideal μ is a normal fuzzy bi-ideal of M if and only if $\mu(0) = 1$.
Here we show that a fuzzy set μ^+ of M such that $\mu^+(x) = \mu(x) + 1- \mu(0)$ for all $x \in M$ is a normal fuzzy bi-ideal of M.

Theorem 1.3.2. Let μ be a fuzzy bi-ideal of M. Let μ^+ be a fuzzy set in M defined by $\mu^+(x) = \mu(x) + 1- \mu(0)$ for all $x \in M$. Then μ^+ is a normal fuzzy bi-ideal of M which contains μ.

Proof: For any $x, y \in M$, we have $\mu^+(x) = \mu(x) + 1 - \mu(0)$ and $\mu^+(0) = 1$ and
$\mu^+(x + y) = \mu(x + y) + 1 - \mu(0) \geq \min(\mu(x), \mu(y)) + 1 - \mu(0) = \min(\mu(x) + 1 - \mu(0), \mu(y) + 1 - \mu(0)) = \min(\mu^+(x), \mu^+(y))$,
Thus $\mu^+(x + y) \geq \min(\mu^+(x), \mu^+(y))$.
Now we have to prove that $\mu^+(x\alpha y) = \min(\mu^+(x), \mu^+(y))$.for all $x, y \in M$ and $\alpha \in\Gamma$.$\mu^+(x\alpha y) = \mu(x\alpha y) + 1 - \mu(0) \geq \min(\mu(x), \mu(y)) + 1 - \mu(0) = \min(\mu(x) + 1 - \mu(0), \mu(y) + 1 - \mu(0)) = \min(\mu^+(x), \mu^+(y))$.

This shows that μ^+ is a fuzzy ideal of M.
Let $x, u, y \in M$ and $\alpha, \beta\in \Gamma$.
$\mu^+(x\alpha u\beta y) = \mu(x\alpha u\beta y) + 1 - \mu(0) \geq \min(\mu(x), \mu(y)) + 1 - \mu(0) = \min(\mu(x) + 1 - \mu(0), \mu(y) + 1 - \mu(0)) = \min(\mu^+(x), \mu^+(y))$.

So μ^+ is a fuzzy bi-ideal of M. Clearly, $\mu \subset \mu^+$ since $\mu^+(x)= \mu(x)+1 - \mu(0) \geq \mu(x)$ as $1 - \mu(0) \geq 0$.

Corollary 1.3.3. Let μ, μ^+ be in Theorem 1.3.2. If there exists $x \in M$ such that $\mu^+(x)= 0$, then $\mu(x) =0$.

Definition 1.3.4. Let N(M) denote the set of all normal fuzzy bi-ideals of M. Note that N(M) is a poset under the set inclusion. A fuzzy set μ in M is called a maximal fuzzy bi-ideal of M if it is non-constant and μ^+ is a maximal element of $(N(M), \subseteq)$.
Following result gives maximal elements when $\mu \in N(M)$ be non-constant.

Theorem 1.3.5. Let μ be a fuzzy bi-ideal of M. Let $f: [0, \mu(0)] \to [0, 1]$ be an increasing fuction. Then the fuzzy set $\mu_f : M \to [0, 1]$ defined by $\mu_f(x) = f(\mu(x))$ is a fuzzy bi-ideal of M. In particular, if $f(\mu(0)) = 1$, then μ_f is a normal; if $f(t) \geq t$ for all $t \in [0, \mu(0)]$, then $\mu \subseteq \mu_f$.
Proof. For any $x, y \in M$,
$\mu_f(x+y) = f(\mu(x+y)) \geq f(\min\{\mu(x), \mu(y)\}) \geq \min\{f(\mu(x)), f(\mu(y))\} = \min\{\mu_f(x), \mu_f(y)\}$,
Let $x, y \in M$ and $\alpha \in \Gamma$.
$\mu_f(x\alpha y) = f(\mu(x\alpha y)) \geq f(\min\{\mu(x), \mu(y)\}) \geq \min\{f(\mu(x)), f(\mu(y))\} = \min\{\mu_f(x), \mu_f(y)\}$,
and for $x, u, y \in M$ and $\alpha, \beta\in \Gamma$.
$\mu_f(x\alpha u\beta y) = f(\mu(x\alpha u\beta y)) \geq f(\min\{\mu(x), \mu(y)\}) \geq \min\{f(\mu(x)), f(\mu(y))\} = \min\{\mu_f(x), \mu_f(y)\}$,
proving that μ_f is a fuzzy bi-ideal of M.
If $f(\mu(0)) = 1$, then $\mu_f(0) = 1$. Thus μ_f is normal.

Assume that $f(t) = f(\mu(x)) \geq \mu(x)$, for any $x \in M$, which gives $\mu \subseteq \mu_f$. Hence the proof. ■

Let N(M) denote the set of all normal fuzzy bi-ideal of M. Note that N(M) is a poset under set inclusion.

Theorem 1.3.6. Let $\mu \in N(M)$ be non-constant such that it is a maximal element of $(N(M), \subseteq)$. Then it takes only two values {0, 1}.

Proof : Since μ is normal, $\mu(0) = 1$. We claim that $\mu(x) = 0$. If not, then there exists $x_0 \in M$ such that $0 < \mu(x_0) < 1$. Define on M a fuzzy set ν by putting

$$\nu(x) = ½ (\mu(x) + \mu(x_0)) \text{ for each } x \in M.$$

Then clearly ν is well-defined and for all x, y ∈ M, we have

$\nu(x + y) = ½\mu(x + y) + ½\mu(x_0) \geq ½(\min (\mu(x), \mu(y)) + ½\mu(x_0) = \min \{½(\mu(x) + \mu(x_0)), ½(\mu(y) + \mu(x_0))\} = \min \{\nu(x),\nu(y)\}$.

Now $\nu(x\alpha y) = ½\mu(x\alpha y) + ½\mu(x_0) \geq ½(\min (\mu(x), \mu(y)) + ½\mu(x_0)) = \min \{½(\mu(x) + \mu(x_0)), ½(\mu(y) + \mu(x_0))\} = \min \{\nu(x),\nu(y)\}$, for all x, y ∈ M and α∈ Γ.

Now $\nu(x\alpha u\beta y) = ½\mu(x\alpha u\beta y) + ½\mu(x_0) \geq ½(\min (\mu(x), \mu(y)) + ½\mu(x_0)) = \min \{½(\mu(x) + \mu(x_0)), ½(\mu(y) + \mu(x_0))\} = \min \{\nu(x),\nu(y)\}$, for all x, u, y ∈ M and α , β∈ Γ.

Thus ν is a fuzzy bi-ideal of M.

Hence ν is a fuzzy bi-ideal of M. By Theorem 1.3.2., ν^+ is a maximal fuzzy bi-ideal of M. Note that $\nu^+(0) = \nu(0) + 1 - \nu(0) = 1$. Thus ν^+ is a normal fuzzy bi-ideal of M and $\nu^+(x_0) = \nu(x_0) + 1 - \nu(0) = ½(\mu(x_0) + \mu(x_0))+1 - ½(\mu(0) + \mu(x_0)) = ½(\mu(x_0) + 1) = \nu(x_0)$, and $\nu (x_0) < 1 = \nu^+(0)$.

Hence ν^+ is a non-constant and μ is not a maximal element of N(M).Which is a contradiction. Thus $\mu(x_0) \geq 0$.

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