International Journal of Latest Research in Science and Technology ISSN (Online):2278-5299
Volume 4, Issue 6: Page No.79-83, November-December 2015 (special Issue Paper)
http://www.mnkjournals.com/ijlrst.htm
Special Issue on International conference on Methematics Sciences-2015(MSC-2015)
Confernece Held at Sadguru Gadage Maharaj College, Karad ,Maharastra,(India)

INTRA-REGULAR Γ-SEMIRINGS

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Abstract- ***In this paper we discuss properties of an intra-regular Γ-semiring. Some characterizations of an intra-regular Γ-semiring by using left ideals, right ideals, ideals, interior-ideals , quasi-ideals and bi-ideals of a Γ-semiring are furnished.***

Key words :- Quasi-ideal, bi-ideal , interior-ideal, intra-regular Γ-semiring.

AMS Mathematics Subject Classification (2010) : 16Y60, 16Y99.

I. INTRODUCTION

A Γ-semiring was introduced by Rao in [10] as a generalization of a semiring. Author studied quasi-ideals in Γ-semirings in [3,4] and gave a definition bi-ideal in [4]. In [7] Lajos considered an intra-regular semigroup and proved some properties of it. Kehayopulu and Tsingelis in [6] proved that the intra-regular ordered semigroups are semilattices of simple semigroups. Some more characterizations of the intra-regular ordered semigroups were discussed by D. Lee and S. Lee in [8]. Shabir , Ali , Batool in [11] gave a definition of an intra-regular semiring and furnished property of it. Author define the notion of an intra-regular Γ-semiring and studied it in [5]. In this paper efforts are made to prove some properties of an intra-regular Γ-semiring. Some more characterizations of an intra-regular Γ-semiring by using left ideals, right ideals, ideals, interior-ideals, quasi-ideals and bi-ideals of a Γ-semiring are studied.

§2. Preliminaries

First we recall some definitions of the basic concepts of Γ-semirings that we need in sequel. For this we follow Dutta and Sardar [1].

Definition 2.1:- *Let S and Γ be two additive commutative semigroups. S is called a Γ-semiring if there exists a mapping $S \times \Gamma \times S \longrightarrow S$ denoted by $a\alpha b$; for all $a, b \in S$ and for all $\alpha \in \Gamma$ satisfying the following conditions:*

(i) $a\alpha(b + c) = (a\alpha b) + (a\alpha c)$
(ii) $(b + c)\alpha a = (b\alpha a) + (c\alpha a)$
(iii) $a(\alpha + \beta)c = (a\alpha c) + (a\beta c)$
(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$; for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

Definition 2.2 :- *An element $0 \in S$ is said to be an absorbing Zero if*
$0\alpha a = 0 = a\alpha 0, a + 0 = 0 + a = a$; *for all $a \in S$ and for all $\alpha \in \Gamma$.*

Definition 2.3:- *A non-empty subset T of a Γ-semiring S is said to be sub-Γ-semiring of S if (T,+) is a subsemigroup of (S,+) and $a\alpha b \in T$; for all $a, b \in T$ and for all $\alpha \in \Gamma$.*

Definition 2.4:- *A non-empty subset T of a Γ-semiring S is called a left (respectively right) ideal of S if T is a subsemigroup of (S,+) and $x\alpha a \in T$ (respectively $a\alpha x \in T$) for all $a \in T, x \in S$ and for all $\alpha \in \Gamma$.*

Definition 2.5 :- *If a non-empty subset T of a Γ-semiring S is both left and right ideal of S, then T is known as an ideal of S.*

Definition 2.6 [3]:- *An additive subsemigroup Q of a Γ-semiring S is a quasi-ideal of S if*
$(S\Gamma Q) \cap (Q\Gamma S) \subseteq Q$.

Definition 2.7 [4]:- *A non-empty subset B of a Γ-semiring S is a bi-ideal of S if B is a sub-Γ-semiring of S and $B\Gamma S\Gamma B \subseteq B$.*

Definition 2.8 [5]:- *An additive subsemigroup I of a Γ-semiring S is an interior-ideal of S $S\Gamma I\Gamma S \subseteq I$.*

Definition 2.9 [2]:- *A proper ideal I of a Γ-semiring S is a completely semiprime if for any $a \in S$, $a\Gamma a \subseteq I$ implies $a \in I$.*

Definition 2.10 [2]:- *An proper ideal I of a Γ-semiring S is semiprime if for any ideal A of S, $A\Gamma A \subseteq I$ implies $A \subseteq I$.*

Theorem 2.11[3]:- For any non-empty subset X of a Γ-semiring S following statements hold. **(I)** $S\Gamma X$ is a left ideal of S. **(II)** $X\Gamma S$ is a right ideal of S. **(III)** $S\Gamma X\Gamma S$ is an ideal of S.

Corollary 2.12[3]:- For any element a of a Γ-semiring S following statements hold.
(I) $S\Gamma a$ is a left ideal of S. **(II)** $a\Gamma S$ is a right ideal of S. **(III)** $S\Gamma a\Gamma S$ is an ideal of S.

Now onwards S denotes a Γ-semiring with an absorbing zero unless otherwise stated.

Theorem 2.13[3] :- Let a be any element of S. Then $(a)_l = N_0 a + S\Gamma a$, $(a)_r = N_0 a + a\Gamma S$ and $(a) = N_0 a + S\Gamma a + a\Gamma S + S\Gamma a\Gamma S$, where N_0 denotes the set of non negative integers.

Theorem 2.14[3]:- Let a be any element of S. Then a quasi-ideal of S generated by a is given by

$(a)_q = N_0 a + (S\Gamma a) \cap (a\Gamma S)$, where N_0 denotes the set of non negative integers.

Theorem 2.15 [4]:- Let a be any element of S. Then a bi-ideal of S generated by a is given by $(a)_b = N_0 a + N_0(a\Gamma a) + a\Gamma S\Gamma a$, where N_0 denotes the set of non negative integers.

Theorem 2.16 :- Let a be any element of S. Then a interior-ideal of S generated by a is $(a)_i = N_0 a + S\Gamma a\Gamma S$, where N_0 denotes the set of non negative integers.

Definition 2.17 [5]:- A Γ-semiring S is said to be an intra-regular Γ-semiring , if for any $x \in S$, $x \in S\Gamma x\Gamma x\Gamma S$.

Example 1:- Consider a set $S = \{0, a, b\}$ and two binary operations $+$ and $\cdot$ are defined on S as follows

+	0	a	b
0	0	a	b
a	a	a	a
b	b	a	b

+	0	a	b
0	0	a	b
a	a	a	a
b	b	a	b

Let $\Gamma = S$. Then both $(S, +)$ and $(\Gamma, +)$ are commutative semigroups. A mapping $S \times \Gamma \times S \to S$ is defined by $x\alpha y = x \cdot \alpha \cdot y$; for all $x, y \in S$, $\alpha \in \Gamma$. Then S forms a Γ-semiring. Here S is an intra-regular Γ-semiring.

Theorem 2.18[5]:- S is intra-regular if and only if each right ideal R and a left ideal L of S satisfy $R \cap L \subseteq L\Gamma R$.

§3. Properties of an Intra-regular Γ-semiring

Properties of an intra-regular Γ-semiring are furnished in the following theorems .

Theorem 3.1:- In an intra-regular Γ-semiring S, an ideal of S is an idempotent ideal.

Proof :- Let S be an intra-regular Γ-semiring and I be an ideal of S. For any $a \in I$, we have $a \in S\Gamma a\Gamma a\Gamma S$. Therefore $a \in S\Gamma a\Gamma a\Gamma S \subseteq S\Gamma I\Gamma I\Gamma S = (S\Gamma I)\Gamma(I\Gamma S) \subseteq I\Gamma I$. This gives $I \subseteq I\Gamma I$. As $I\Gamma I \subseteq I$ holds always, we get $I\Gamma I = I$. Therefore an ideal of S is an idempotent ideal.

]Theorem 3.2:- If B is an ideal of an ideal of S , then $(B)^3 = (B)\Gamma(B)\Gamma(B) \subseteq B$.

Proof :- Let B be an ideal of an ideal A of S. Therefore $(B)\Gamma(B)\Gamma(B) \subseteq A\Gamma(B)\Gamma A = A\Gamma(B + B\Gamma S + S\Gamma B + S\Gamma B\Gamma S)\Gamma A = (A\Gamma B + A\Gamma B\Gamma S + A\Gamma S\Gamma B + A\Gamma S\Gamma B\Gamma S)\Gamma A \subseteq (B + B\Gamma S + A\Gamma B + A\Gamma B\Gamma S)\Gamma A \subset (B + B\Gamma S + B + B\Gamma S)\Gamma A \subseteq B + B\Gamma A + B + B\Gamma A \subseteq B$

. Thus we get

$(B)^3 = (B)\Gamma(B)\Gamma(B) \subseteq B$.

■

Remark :- An ideal of an ideal of a Γ-semiring S need not be an ideal of S.

Example 2:- Let $S = \{0,1,2,3\}$. Two binary operations + and · are defined on S as follows.

.	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	1
3	0	0	1	2

+	0	1	2	3
0	0	0	0	0
1	0	1	0	0
2	0	0	2	0
3	0	0	0	3

If $\Gamma = S$, then $(S, +)$ and $(\Gamma, +)$ both are commutative semigroups. A mapping $S \times \Gamma \times S \to S$ is defined by $x\alpha y = x \cdot y$; for all x, y in S and $\alpha \in \Gamma$. Then S forms a Γ-semiring. Here $I = \{0,1,2\}$ is an ideal of S. $B = \{0,2\}$ is a two sided ideal of an ideal I of S. But B is not an ideal of S since $3\alpha 2 = 1 \notin B$, for all $\alpha \in \Gamma$.But in an intra-regular Γ-semiring S we have Theorem 3.3

Theorem 3.3:- In an intra-regular Γ-semiring S, an ideal of an ideal of S is an ideal of S.

Proof :- Let S be an intra-regular Γ-semiring , A be an ideal of S and B be an ideal of A. Hence by Theorem 3.2, we have $(B)^3 = (B)\Gamma(B)\Gamma(B) \subseteq B$. By Theorem 3.1, we have any ideal of S is an idempotent ideal. Therefore $(B)^3 = (B)\Gamma(B)\Gamma(B) = (B)\Gamma(B) = (B)$. Thus $(B) \subseteq B$. As $B \subseteq (B)$ holds always, we get $(B) = B$. This shows that B is an ideal of S.

Theorem 3.4:- If S is intra-regular, then any proper ideal of S is semiprime.

Proof :- Let S be an intra-regular Γ-semiring and P be a proper ideal of S. Let A be any ideal of S such that $A\Gamma A \subseteq P$. For any $a \in A$, we have $a \in S\Gamma a\Gamma a\Gamma S$. Hence we have $S\Gamma a\Gamma a\Gamma S \subset S\Gamma A\Gamma A\Gamma S = (S\Gamma A)\Gamma(A\Gamma S) \subset A\Gamma A \subset P$. Therefore $a \in P$. Thus $a \in A$ implies $a \in P$. This shows that $A \subseteq P$. Therefore P is a semiprime ideal of S.

Theorem 3.5:-If S is intra-regular, then a proper interior-ideal of S is semiprime.

Proof of following theorem is straightforward so omitted.

Theorem 3.6 :- If $S\Gamma a = S$ or $a\Gamma S = S$ holds for all $a \in S$, then S is intra-regular .

§4. Characterizations of an Intra-regular Γ-semiring

Various characterizations of an intra-regular Γ-semiring are discussed in this section.

Theorem 4.1 :- In S following statements are equivalent.

(1) S is intra-regular.

(2) Each ideal of S is completely semiprime.

(3) $a \in (a\Gamma a)$, for any $a \in S$.

(4) $(a) = (a\Gamma a)$, for any $a \in S$.

Proof :- **(1) ⇒ (2)** Let P be a proper ideal of S. For a be any element of S, $a\Gamma a \subseteq P$. As $a \in S$ and S is intra-regular, we have $a \in S\Gamma a\Gamma a\Gamma S$. Therefore $S\Gamma a\Gamma a\Gamma S \subset S\Gamma P\Gamma S \subset P$. Hence $a \in P$. Thus $a\Gamma a \subseteq P$ implies $a \in P$. Therefore P is a completely semiprime ideal of S.

(2) ⇒ (1) Let $a \in S$. We have $S\Gamma a\Gamma a\Gamma S$ is an ideal of S. By assumption $S\Gamma a\Gamma a\Gamma S$ is a completely semiprime ideal of S. Therefore $(a\Gamma a)\Gamma(a\Gamma a) \subseteq S\Gamma a\Gamma a\Gamma S$ implies $a\Gamma a \subseteq S\Gamma a\Gamma a\Gamma S$. Hence $a \in S\Gamma a\Gamma a\Gamma S$. This shows that S is an intra-regular Γ-semiring.

(2) ⇒ (3) Let $a \subset S$. By assumption $(a\Gamma a)$ is a completely semiprime ideal of S. We have $a\Gamma a \subset (a\Gamma a)$ always. Therefore $a \in (a\Gamma a)$, for any $a \in S$.

(3) ⇒ (4) Let $a \in S$. By (3), $a \in (a\Gamma a)$. Therefore $(a) \subseteq (a\Gamma a)$. Now

$$(a\Gamma a) = N_0(a\Gamma a) + S\Gamma(a\Gamma a) + (a\Gamma a)\Gamma S + S\Gamma(a\Gamma a)\Gamma S \subseteq N_0(S\Gamma a) + S\Gamma a + a\Gamma S + S\Gamma a\Gamma S \subseteq S\Gamma a + a\Gamma S + S\Gamma a\Gamma S \subseteq (a)$$

. Hence $(a) = (a\Gamma a)$.

(4) ⇒ (2) Let P be a proper ideal of S. For a be any element of S, $a\Gamma a \subseteq P$. By (4), we have $(a) = (a\Gamma a)$. Hence $a\Gamma a \subseteq P$ implies $(a\Gamma a) \subseteq P$. Therefore $(a) \subseteq P$. Hence $a \in P$. Therefore P is a completely semiprime ideal of S.

Theorem 4.2:-S is intra-regular if and only if each interior-ideal of S is completely semiprime.

Proof :- Let S be an intra-regular Γ-semiring and P be a proper interior-ideal of S. For a be any element of S, $a\Gamma a \subseteq P$. Then we have $a \in S\Gamma a\Gamma a\Gamma S$. Therefore $S\Gamma a\Gamma a\Gamma S \subseteq S\Gamma P\Gamma S \subseteq P$. Hence $a \in P$. Therefore P is a completely semiprime interior-ideal of S.

Conversely, assume that each interior-ideal of S is completely semiprime. Let $a \in S$. We have $S\Gamma a\Gamma a\Gamma S$ is an interior-ideal of S. Therefore by assumption $S\Gamma a\Gamma a\Gamma S$ is completely semiprime. Hence $(a\Gamma a)\Gamma(a\Gamma a) \subseteq S\Gamma a\Gamma a\Gamma S$ implies $a\Gamma a \subseteq S\Gamma a\Gamma a\Gamma S$. Hence $a \in S\Gamma a\Gamma a\Gamma S$. This shows that S is an intra-regular Γ-semiring.

Corollary 4.3:-S is intra-regular if and only if each ideal of S is completely semiprime.

Theorem 4.4:- In S following statements are equivalent.

(1) S is intra-regular.

(2) $(x)_b \subseteq S\Gamma(x)_b\Gamma(x)_b\Gamma S$

(3)

$$(x)_b \cap (x)_q \subseteq (S\Gamma(x)_b\Gamma(x)_q\Gamma S) \cap (S\Gamma(x)_q\Gamma(x)_b\Gamma S)$$

(4) $(x)_q \subseteq S\Gamma(x)_q\Gamma(x)_q\Gamma S$

Proof :- **(1) ⇒ (2)** Let $a \in (x)_b$. Therefore $a \in S\Gamma a\Gamma a\Gamma S \subseteq S\Gamma(x)_b\Gamma(x)_b\Gamma S$. Hence $(x)_b \subseteq S\Gamma(x)_b\Gamma(x)_b\Gamma S$.

(2) ⇒ (4) Implication holds as every quasi-ideal is a bi-ideal.

(4) ⇒ (1) Let $x \in S$. Therefore by assumption, $(x)_q \subseteq S\Gamma(x)_q\Gamma(x)_q\Gamma S$. Now

$$S\Gamma(x)_q\Gamma(x)_q\Gamma S = (S\Gamma(N_0x + (S\Gamma x) \cap (x\Gamma S)))\Gamma((N_0x + (S\Gamma x) \cap (x\Gamma S))\Gamma S) \subseteq (S\Gamma x)\Gamma(x\Gamma S)$$

. Hence $x \in (x)_q \subseteq S\Gamma x\Gamma x\Gamma S$. Therefore S is intra-regular.

(1) ⇒ (3) Assume that S is an intra-regular Γ-semiring. Let $a \in (x)_b \cap (x)_q$. Therefore $a \in S\Gamma a\Gamma a\Gamma S \subseteq S\Gamma(x)_b\Gamma(x)_q\Gamma S$ and $\in S\Gamma a\Gamma a\Gamma S \subseteq S\Gamma(x)_q\Gamma(x)_b\Gamma S$. Hence

$$(x)_b \cap (x)_q \subseteq (S\Gamma(x)_b\Gamma(x)_q\Gamma S) \cap (S\Gamma(x)_q\Gamma(x)_b\Gamma S)$$

.

(3) ⇒ (1) Let $x \in S$. Therefore by assumption,

$$(x)_b \cap (x)_q \subseteq (S\Gamma(x)_b\Gamma(x)_q\Gamma S) \cap (S\Gamma(x)_q\Gamma(x)_b\Gamma S)$$

. Now

$$S\Gamma(x)_b\Gamma(x)_q\Gamma S = (S\Gamma(N_0x + N_0(x\Gamma x) + x\Gamma S\Gamma x))\Gamma((N_0x + (S\Gamma x) \cap (x\Gamma S))\Gamma S) \subseteq (S\Gamma x)\Gamma(x\Gamma S)$$

. Hence $x \in (x)_b \cap (x)_q \subseteq S\Gamma x\Gamma x\Gamma S$. Therefore S is intra-regular.

Theorem 4.5 :- In S following statements are equivalent.

(1) S is intra-regular.

(2) $(x)_l \cap (x)_b \subseteq (x)_l\Gamma(x)_b\Gamma S$

(3) $(x)_l \cap (x)_q \subseteq (x)_l\Gamma(x)_q\Gamma S$

(4) $(x)_r \cap (x)_b \subseteq S\Gamma(x)_b\Gamma(x)_r$

(5) $(x)_r \cap (x)_q \subseteq S\Gamma(x)_q\Gamma(x)_r$

Proof :- **(1) ⇒ (2)** Let $a \in (x)_l \cap (x)_b$. Therefore $a \in S\Gamma a\Gamma a\Gamma S \subseteq S\Gamma(x)_l\Gamma(x)_b\Gamma S$. Hence $(x)_l \cap (x)_b \subseteq (x)_l\Gamma(x)_b\Gamma S$.

(2) ⇒ (3) Implication holds as every quasi-ideal is a bi-ideal.

(3) ⇒ (1) Let $x \in S$. Therefore by assumption, $(x)_l \cap (x)_q \subseteq (x)_l\Gamma(x)_q\Gamma S$. Now

$$(x)_l\Gamma(x)_q\Gamma S = (N_0x + (S\Gamma x))\Gamma((N_0x + (S\Gamma x) \cap (x\Gamma S))\Gamma S) \subseteq (S\Gamma x)\Gamma(x\Gamma S)$$

. Hence $x \in (x)_q \subseteq S\Gamma x\Gamma x\Gamma S$. Therefore S is intra-regular.

(1) ⇒ (4) Assume that S is an intra-regular Γ-semiring. Let $a \in (x)_r \cap (x)_b$.

Therefore $a \in S\Gamma a\Gamma a\Gamma S \subseteq S\Gamma(x)_b\Gamma(x)_r$ and $a \in S\Gamma a\Gamma a\Gamma S \subseteq S\Gamma(x)_b\Gamma(x)_r\Gamma S \subseteq S\Gamma(x)_b\Gamma(x)_r$. Hence $(x)_r \cap (x)_b \subseteq S\Gamma(x)_b\Gamma(x)_r$.

(4)⇒(5) Implication holds as every quasi-ideal is a bi-ideal.

(5)⇒(1) Let $x \in S$. Therefore by assumption, $(x)_r \cap (x)_q \subseteq S\Gamma(x)_q\Gamma(x)_r$. Now

$$S\Gamma(x)_q\Gamma(x)_r = (S\Gamma(N_0x + (S\Gamma x)))\Gamma(N_0x + (S\Gamma x) \cap (x\Gamma S)) \subseteq (S\Gamma x)\Gamma(x\Gamma S)$$

.

Hence $x \in (x)_r \cap (x)_q \subseteq S\Gamma x\Gamma x\Gamma S$. Therefore S is intra-regular.

Theorem 4.6 :- In S following statements are equivalent.

(1) S is intra-regular.

(2) $(x)_i \cap (x)_b \cap (x)_r \subseteq (x)_i\Gamma(x)_b\Gamma(x)_r$

(3) $(x) \cap (x)_b \cap (x)_r \subseteq (x)\Gamma(x)_b\Gamma(x)_r$

(4) $(x)_i \cap (x)_q \cap (x)_r \subseteq (x)_i\Gamma(x)_q\Gamma(x)_r$

(5) $(x) \cap (x)_q \cap (x)_r \subseteq (x)\Gamma(x)_q\Gamma(x)_r$

Proof :- **(1) ⇒ (2)** Let $a \in (x)_i \cap (x)_b \cap (x)_r$. Therefore

$$a \in S\Gamma a\Gamma a\Gamma S \subseteq (S\Gamma a\Gamma S)\Gamma(a\Gamma S\Gamma a)\Gamma(a\Gamma S) \subseteq (S\Gamma(x)_i\Gamma S)\Gamma((x)_b\Gamma S\Gamma(x)_b)\Gamma((x)_r\Gamma S) \subseteq (x)_i\Gamma(x)_b\Gamma(x)_r$$

. Hence $(x)_i \cap (x)_b \cap (x)_r \subseteq (x)_i\Gamma(x)_b\Gamma(x)_r$.

(2) ⇒ (3) and **(4) ⇒ (5)**

As every ideal is an interior ideal, implications follow.

(2) ⇒ (4) and **(3) ⇒ (5)**

As every quasi-ideal is a bi-ideal, implications follow.

(5) ⇒ (1) Let $x \in S$. Therefore by assumption, $(x) \cap (x)_q \cap (x)_r \subseteq (x)\Gamma(x)_q\Gamma(x)_r$. Now

$$(x)\Gamma(x)_q\Gamma(x)_r = (N_0x + S\Gamma x + x\Gamma S + S\Gamma x\Gamma S)\Gamma(N_0x + (S\Gamma x) \cap (x\Gamma S))\Gamma(N_0x + x\Gamma S) \subseteq (S\Gamma x)\Gamma(x\Gamma S)$$

. Hence $x \in S\Gamma x\Gamma x\Gamma S$. Therefore S is intra-regular.

Theorem 4.7:- In S following statements are equivalent.

(1) S is intra-regular.

(2) $(x)_i \cap (x)_b \cap (x)_l \subseteq (x)_l\Gamma(x)_b\Gamma(x)_i$

(3) $(x) \cap (x)_b \cap (x)_l \subseteq (x)_l\Gamma(x)_b\Gamma(x)$

(4) $(x)_i \cap (x)_q \cap (x)_l \subseteq (x)_l\Gamma(x)_q\Gamma(x)_i$

(5) $(x) \cap (x)_q \cap (x)_r \subseteq (x)_l\Gamma(x)_q\Gamma(x)$

Proof :- (1) ⇒ (2) Let $a \in (x)_i \cap (x)_b \cap (x)_i$. Therefore

$a \in S\Gamma a\Gamma a\Gamma S \subseteq (S\Gamma S\Gamma a)\Gamma a\Gamma(S\Gamma a\Gamma S) \subseteq (S\Gamma S\Gamma(x)_i)\Gamma(x)_b\Gamma(S\Gamma(x)_i\Gamma S) \subseteq (x)_l\Gamma(x)_b\Gamma(x)_i$

.

Hence $(x)_i \cap (x)_b \cap (x)_l \subseteq (x)_l\Gamma(x)_b\Gamma(x)_i$.

(2) $\Rightarrow$ **(3)** and **(4)** $\Rightarrow$ **(5)**

As every ideal is an interior ideal, implications follow.

(2) $\Rightarrow$ **(4)** and **(3)** $\Rightarrow$ **(5)**

As every quasi-ideal is a bi-ideal, implications follow.

(5) $\Rightarrow$ **(1)** Let $x \in S$. Therefore by assumption, $(x) \cap (x)_q \cap (x)_l \subseteq (x)_l\Gamma(x)_q\Gamma(x)$. Now $(x)_l\Gamma(x)_q\Gamma(x) = (N_0x + S\Gamma x)\Gamma(N_0x + (S\Gamma x) \cap (x\Gamma S))\Gamma(N_0x + S\Gamma x + x\Gamma S + S\Gamma x\Gamma S) \subseteq (S\Gamma x)\Gamma(x\Gamma S)$
Hence $x \in S\Gamma x\Gamma x\Gamma S$. Therefore S is intra-regular.

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