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WEYL GROUP OF SPECIAL LINEAR ALGEBRA

Subhash M Gaded,
Assistant Professor,
R. K. Talreja College of Arts, Science & Commerce,
Ulhasnagar-03, Dist. Thane, Maharashtra,
Email: gadedsubhash@gmail.com

Abstract- ***In this paper we discuss about the Root space decomposition of Special Linear Algebra. We show that the Weyl group of Special Linear Algebra sl(n, F) is the permutation group on n symbols. The weyl group of A_3 is calculated.***

Keywords: Special linear algebra, Root space decomposition, Weyl group.
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I. INTRODUCTION

Definition (Lie algebra). A Lie algebra is a vector space L over a field F, with an operation $[,]: L\times L\to L$, denoted $(x, y) \mapsto [x, y]$,(called the bracket or commutator of x and y), satisfying the following properties :

(L1) The bracket operation is bilinear.
(L2) $[x, x] = 0$, for all $x\in L$.
(L3) $[x, [y, z]] + [y, [z, x]] + [z,[x, y]] = 0$, for all $x, y, z \in L$.

(L3) is called the Jacobi identity.
$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x]$
Hence, condition (L1) and (L2) implies
(L2') $[x, y] = -[y, x]$ (anticommutativity), for all $x, y \in L$.
If char $F\neq 2$, then putting $x = y$ in (L2'), shows that (L2') implies (L2).

Lie Subalgebra: A subspace K of a Lie algebra L is called a subalgebra if $[x, y] \in K$, whenever $x, y \in K$.
Unless specifically stated, we shall be concerned with Lie algebras L whose underlying vector space is finite dimensional.
Some Examples:

(1) Any vector space V, with $[x, y] = 0$, for all $x, y \in V$ is a Lie algebra called Abelian Lie algebra. In particular, the field F may be regarded as a 1-dimensional abelian Lie algebra.

(2) Let V be a finite dimensional vector space over F with $\dim(V) = n$. Let End V be the set of all linear transformations from $V \to V$. This is again a vector space over F of dimension n^2.
Define an operation on End V, by $[x, y] = xy - yx$.
With this operation End V becomes a lie algebra over F.$[x_1 + x_2, y] = (x_1 + x_2)y - y(x_1 + x_2)$
$= x_1y + x_2y - yx_1 - yx_2$
$= (x_1y - yx_1) + (x_2y - yx_2)$
$= [x_1, y] + [x_2, y]$ for all $x_1, x_2, y \in End\ V$.
Similarly, $[x, y_1+y_2] = [x, y_1] + [x, y_2]$, for all $x, y_1, y_2\in End\ V$.
(L2) $[x, x] = xx - xx = 0$, for all $x \in End\ V$.
(L3)$[x, [y, z]] + [y, [z, x]] + [z, [x, y]]$
$= [x, (yz- zy)] + [y, (zx- xz)] + [z, (xy- yx)]$
$= (x(yz- zy) - (yz- zy)x) + (y(zx- xz) - (zx- xz)y) + (z(xy- yx) - (xy- yx)z)$
$= xyz - xzy - yzx + zyx + yzx - yxz - zxy + xzy + zxy - zyx - xyz + yxz$
$= 0$:

$End\ V$ (also written $gl(V)$) is called General linear algebra. Any subalgebra of a Lie algebra $gl(V)$ is called a linear Lie algebra. $gl(V)$ can be identified with the set of all $n \times n$ matrices over F, denoted $gl(n, F)$, with the Lie bracket defined by
$[x, y] = xy - yx$
wherexy is the usual product of the matrices x and y. As a vector space, $gl(n, F)$ has a basis consisting of the matrix units e_{ij} for $1 \leq i, j\leq$ n. Here, e_{ij} is the $n \times n$ matrix which has 1 in the$i$ - j position and 0 elsewhere.
As$e_{ij}e_{kl} = \delta_{jk}e_{il}$
It follows that:
$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}$.
whereδ is the kronecker delta, defined by,
$\delta_{ij} = 1$, if $i = j$ and
$\delta_{ij} = 0$, if $i\neq j$.

Classical Lie algebras *An*

(2) An: Let dim $V = n + 1$. Denote the set of all endomorphism's of V having trace zero, by $sl(V)$ or $sl(n + 1, F)$. (Trace of a square matrix is the sum of its diagonal entries). Since $Tr(xy) = Tr(yx)$ and $Tr(x + y) = Tr(x) + Tr(y)$, $sl(V)$ is a subalgebra of $gl(V)$, called the **Special Linear algebra**.$sl(V)$ is a proper subalgebra of $gl(V)$. $sl(V)$ has a basis consisting of the e_{ij} for $i\neq j$ together with $e_{ii} - e_{i+1,i+1}$ for $1 \leq i\leq n$.

Adjoint representation:

Themap ad:L→DerLsending$x$toadxiscalledadjointrepresentationofL.

Derivation: AderivationofVisalinear map$D$: V × V → V such that $D(ab) = aD(b) +D(a)b$,foralla,$b\in V$. Let $Der\ V$be the set of derivations of V. This set isclosedunder addition and scalar multiplication and

containsthezero map. Hence, $Der\ V$ is a vector subspace of $gl(V)$.

Ideals:A subspace I of a Lie algebra L iscalled an ideal of L if $[x, y] \in I$,for all $x \in L$, $y \in I$.

Derived series: A derived series of a LieAlgebra L is a sequence of ideals of L defined by $L^{(0)} = L, L^{(1)} = [L, L]$, $L^{(2)} = [L^{(1)}, L^{(1)}], \cdots, L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$.

Solvable: A Lie Algebra L is called Solvableif $L^{(n)} = 0$ for some n.

Radical: Let L be an arbitrary Lie algebraand let S be a maximal solvable ideal.If I is any otherSolvable ideal of L, then S+ I = S. By maximality of S, we get S + I = S, or I ⊂S. This proves the existence of a unique maximal solvableideal, called the radical of L and denoted Rad L.

Semisimple: A Lie algebra L is called semisimpleif Rad $L = 0$.

RootSystem: AsubsetΦofaEuclideanspace$E$iscalledarootsysteminEifitsatisfiesthefollowingaxioms:

(R1)Φisfinite,spansE,anddoesnotcontain0.

(R2)If$\alpha \in \Phi$,theonlymultiplesofαinΦare$\pm\alpha$.

(R3) If $\alpha \in \Phi$, the reflection $\sigma_\alpha$permutes the elementsofΦ.

(R4) If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

Rank. The dimension of E is called therankof the root systemΦ.

ThebaseforaRootsystemandWeylgroup:

Base: A subset Δ of Φ ⊆V is called abaseif,

(B1) Δ is a basis ofV,

(B2) Each root β in Φ can be writtenas,

$$\beta = \sum_{\alpha \in \Delta} \kappa_\alpha \alpha$$

Wherethecoefficients$k_\alpha$areeitherallnonnegativeintegersorallnon-positiveintegers.TherootsinΔarecalled**simple roots**.

The Weyl group of a root system:

For each root $\alpha \in \Phi$ the reflection σ_α is an invertible lineartransformation on E. The subgroup of $GL(E)$ of invertiblelinear transformations of E generated by the reflections σ_α ($\alpha \in \Phi$) is known as the Weyl group ofΦ, denoted by W.

Lemma: The Weyl group W associated to Φfinite.

Theorem: Every root system Φ has abase.

Weyl chamber: The Weyl chambers aredefined to be the components of the complement in E ofthe union of all hyperplanes perpendicular to the roots.Each regular $\gamma \in E$, therefore belongs to precisely one(connected component)weyl chamber of $E \setminus \bigcup_\alpha P_\alpha$, denoted$\mathfrak{C}(\gamma)$. The elements of weyl group W are orthogonal andpermute the roots. Therefore, the weyl group W permutesthe Weyl chambers.

Root space decomposition

Asubalgebra is called toral if it consists of semisimple elements.Any toralsubalgebrais abelian by the following reasoning. Let T be toral,$x \in T$, so adx is semisimple and so over an algebraicallyclosed F it is diagonalizable. So if $ad\ x$ has only 0 eigenvalues,then $ad\ x = 0$. Suppose it has an eigenvalue $a \neq 0$, i.e. there is a $y \in T$, such that $[x, y] = ay$. Since y isalso semisimple, so is $ad\ y$ and it has linearly independenteigenvectors $y_1 = y, \cdots, y_n$(since $ad(y)(y) = 0$) of eigenvalues0, $b_2, \cdots, b_n$, we can write x in this basis as $x = a_1 y_1 + \cdots + a_n y_n$. Then $-a_y = ady(x) = 0.y + b_2 a_2 y_2 + \cdots$, i.e. y is a linear combination of the other eigenvectors,which is impossible. So $a = 0$ and $ad\ x = 0$ for all $x \in T$,i.e. $[x, y] = 0$ for all $x, y \in T$.

Let H be a maximal toralsubalgebra of L, i.e. not includedin any other. For any $h_1, h_2 \in H$, we have $adh_1 \circ ad\ h_2(x) = [h_1, [h_2, x]] = -[h_2, [x, h_1]] - [x, [h_1, h_2]] = [h_2, [h_1, x]] = ad\ h_2 \circ ad\ h_1(x)$ by the Jacobi identity, so $adL\ H$consists of commuting semisimple endomorphism's and by astandard theorem in linear algebra these are simultaneouslydiagonalizable . So we can find Eigen spaces $L\alpha = \{x \in L \mid [h,x] = \alpha(h)x$ for all $h \in H\}$ for $\alpha \subset H*$, such that theyform a basis of L, i.e. we can write the**Root space decomposition**

(or Cartan decomposition)

$$\boldsymbol{L = C_L(H) \oplus \coprod_{\alpha \in \Phi} L_\alpha}$$

Theorem: Let H be a maximal toralsubalgebra of$L$. Then $H = C_L(H)$.

Root Space Decomposition of $sl(n + 1, F)$: Let H consist of the diagonal matrices in the algebra $sl(n + 1, F)$ of all $(n + 1) \times (n+ 1)$ matrices with trace zero.

$$H=\{\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n+1} \end{pmatrix} | a_1 + a_2 + \cdots + a_{n+1} = 0\}$$

As a basis of H we take:

$$h_1 = \begin{pmatrix} 1 & 0 & 0 & & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$h_2 = \begin{pmatrix} 0 & 0 & 0 & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$\vdots$

$$h_n = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}$$

Let $\epsilon_i \in H*$denote the linear function which assigns to eachdiagonal matrix its i^{th} (diagonal) entry, i.e. if $h = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n+1} \end{pmatrix} \in H$

$\epsilon_i(h) = a_i[h, e_{ij}] = he_{ij} - e_{ij}h = a_i e_{ij} - a_j e_{ij}$

$= \epsilon_i(h)e_{ij} - \epsilon_j(h)e_{ij} = (\epsilon_i - \epsilon_j)(h)e_{ij}$

$\therefore [h, e_{ij}] = (\epsilon_i - \epsilon_j)(h)e_{ij}$, for all $h \in H$.

Thus, we see that e_{ij} is a joint eigenvector for the maps *ad* (*h*) with $h \in H$ and with eigenvalue $\epsilon_i - \epsilon_j \in H^*$. Thisshows that *H* is a maximal commutative subalgebra of *sl*(*n + 1, F*). Any $x \in sl(n+1, F)$ can be written as the sumof some element $h' \in H$ and the matrices e_{ij} with $i \neq j$.Vanishing of [*h*, *x*], for all $h \in H$ then immediately impliesthat all the coefficients of the e_{ij} have to vanish. On theother hand, for any element $h \in H$ the map *ad* (*h*) :*sl*(*n + 1, F*) →*sl*(*n + 1, F*) is diagonalizable, so *H* is a MaximalToralsubalgebra of *sl*(*n + 1, F*).

The linear functions of the form $\Phi = \{\epsilon_i - \epsilon_j : i \neq j\}$ are the roots of *sl*(*n + 1, F*) relative to H.Each of the root spaces $sl(n+1, F)_{\epsilon_i - \epsilon_j} = \{x \in sl(n+1, F) \mid [h, x] (\epsilon_i - \epsilon_j)(h)x\}$is one dimensional and spanned by e_{ij}.

The Root Space Decomposition (or Cartan decomposition)of *sl*(*n + 1, F*) is given by,

$$sl(n+1, F) = H \oplus \oplus_{i>j}(Fe_{ij} \oplus F\, e_{ji})$$

Cartan matrix of *sl*(*n + 1, F*):

The base for *sl*(*n + 1, F*) is given by $\{\alpha_i = \epsilon_i - \epsilon_{i+1}\}$. Thismeans that for any two consecutive simple roots we have $< \alpha_i, \alpha_{i+1} > = < \alpha_{i+1}, \alpha_i > = -1$, $< \alpha_i, \alpha_i > = 2$, and $< \alpha_i, \alpha_j > = 0$ otherwise. This gives the following Cartan matrix oftype A_n

.

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

Thus the Cartan matrix of*sl*(*n + 1, F*) has each entry onthe main diagonal equal to 2, in the two diagonals aboveand below the main diagonal all entries equal -1, while allother entries are zero.

Weyl group of A_n

The Weyl group is generated by reflections in the hyper planesorthogonal to the roots and thus consist only of orthogonaltransformations. Consider the orthonormal basisof the root space such that the coordinates of any root areintegers between -2 and 2. The simple roots of $A_n$are givenby $\alpha_i = \{e_i - e_{i+1}: 1 \leq i \leq n\}$Consider the reflection$\sigma_{\alpha_i}$:

$\sigma_{\alpha_i}(e_i)$

$= \sigma_{e_i - e_{i+1}}(e_i)$

$= e_i - \frac{2(e_i,\ e_i - e_{i+1})}{(e_i - e_{i+1}, e_i - e_{i+1})}(e_i - e_{i+1})$

$= e_i - \frac{2((e_i, e_i) - (e_i, e_{i+1}))}{(e_i, e_i) - (e_i, e_{i+1}) - (e_{i+1} e_i) - (e_{i+1}, e_{i+1})}(e_i - e_{i+1})$

$= e_i - \frac{2(1-0)}{(1-0-0+1)}(e_i - e_{i+1})$

$= e_i - (e_i - e_{i+1})$

$= e_{i+1}$

$\sigma_{\alpha_i}(e_{i+1})$

$= \sigma_{e_i - e_{i+1}}(e_{i+1})$

$= e_{i+1} - \frac{2(e_{i+1}, e_i - e_{i+1})}{(e_i - e_{i+1}, e_i - e_{i+1})}(e_i - e_{i+1})$

$= e_{i+1} - \frac{2((e_{i+1}, e_i) - (e_{i+1}, e_{i+1}))}{(e_i, e_i) - (e_i, e_{i+1}) - (e_{i+1} e_i) - (e_{i+1}, e_{i+1})}(e_i - e_{i-1})$

$= e_{i+1} - \frac{2(0-1)}{(1-0-0+1)}(e_i - e_{i+1})$

$= e_{i+1} + (e_i - e_{i+1})$

$= e_i$

For $j \neq i, i+1$

$\sigma_{\alpha_i}(e_j)$

$= \sigma_{e_i - e_{i+1}}(e_j)$

$= e_j - \frac{2(e_j,\ e_i - e_{i+1})}{(e_i - e_{i+1}, e_i - e_{i+1})}(e_i - e_{i+1})$

$= e_j - \frac{2((e_j, e_i) - (e_j, e_{i+1}))}{(e_i, e_i) - (e_i, e_{i+1}) - (e_{i+1} e_i) - (e_{i+1}, e_{i+1})}(e_i - e_{i+1})$

$= e_j - \frac{2(0-0)}{(1-0-0+1)}(e_i \quad e_{i+1})$

$= e_j$

Therefore, $\sigma_{\alpha i}$maps

$e_i \mapsto e_{i+1}$

$e_{i+1} \mapsto e_i$

$e_j \mapsto e_j$ for $j \neq i$, $i+1$.

Thus, σ_{α_i}generate all possible permutations of the *n* + 1coordinates. We thus see that the weyl group of *sl*(*n*+ 1,*F*) is the permutation group S_{n+1}on *n* + 1 symbols.

Weyl group of $A3$

Therootsystemof*sl*(*n*+ 1,*F*)consistsofvectorsoftheform$\alpha_{kl} = e_{kk} - e_{ll}$. To simplify calculations, identify each$e_{kk}$with unit vectorϵ_k.Wenowhavethe set of roots$\{\epsilon_i - \epsilon_j | i \neq j\}$.The simple rootsare $\alpha_i = \epsilon_i \quad \epsilon_{i+1}$.

Then for *A*3 we have theroot s y s t e m ,

$\Phi = \{\pm\alpha_1 = \pm(1, -1, 0, 0), \pm\alpha_2 = \pm(0, 1, -1, 0), \pm\alpha_3 = \pm(0, 0, 1, -1),$
$\pm\alpha_4 = \pm(1, 0, -1, 0), \pm\alpha_5 = \pm(1, 0, 0, -1), \pm\alpha_6 = \pm(0, 1, 0, -1)\}.$

The baseis

$\Delta = \{\alpha_1 = (1, -1, 0, 0), \alpha_2 = (0, 1, -1, 0), \alpha_3$

$= 0,1,-1)\}.$

To find the element of the Weyl group generated by $\alpha_1 = (1, -1, 0, 0)$ we calculate $\sigma_{\alpha_1}$for each of the roots inΦ.

$\sigma_{\alpha_1}(\alpha_1)$=-$\alpha_1\sigma_{\alpha_1}$(-$\alpha_1$)

=$\alpha_1.\sigma_{\alpha_1}(\alpha_2)$=$\sigma_{\alpha_1}((0,1,-1,0))$=(0,1,1)–

$$2\frac{<(1,-1,0,0),(0,1,-1,0)>}{<(0,1,-1,0)(0,1,-1,0)>}(1,-1,0,0)$$

=(0,1,−1,0)+(1,−1,0,0)

=(1,0,−1,0)

=α_4

Similar calculationsshow:

$\sigma\alpha 1(-\alpha\_2)=-\alpha 4$ $\sigma\alpha 1(\pm\alpha 3)=\pm\alpha 3$ $\sigma\alpha 1(\pm\alpha 4)=\pm\alpha 2$
$\sigma\alpha 1(\pm\alpha 5)=\pm\alpha 6$ $\sigma\alpha 1(\pm\alpha 6)=\pm\alpha 5$.
ThisgivesoneelementoftheWeylgroup,namelythe permutation

$\{\pm\alpha_1 \rightleftharpoons \mp\alpha_1, \pm\alpha_2 \rightleftharpoons \pm\alpha_4,, \pm\alpha_3 \rightleftharpoons \pm\square_3, \ ,\pm\square_4 \rightleftharpoons \pm\square_2, \pm\square_5 \rightleftharpoons \pm\square_6, ,\pm\square_6 \rightleftharpoons \pm\square_5\}$.

Table below shows the permutations with respect toallsimpleroots.
Weyl group of$A3$:

Reflection	$\pm\alpha_1$	$\pm\alpha_2$	$\pm\alpha_3$	$\pm\alpha_4$	$\pm\alpha_5$	$\pm\alpha_6$
$\sigma\alpha_1$	$\mp\alpha_1$	$\pm\alpha_4$	$\pm\alpha_3$	$\pm\alpha_2$	$\pm\alpha_6$	$\pm\alpha_5$
$\sigma\alpha_2$	$\pm\alpha_4$	$\mp\alpha_2$	$\pm\alpha_6$	$\pm\alpha_1$	$\pm\alpha_5$	$\pm\alpha_3$
$\sigma\alpha_3$	$\pm\alpha_1$	$\pm\alpha_6$	$\mp\alpha_3$	$\pm\alpha_5$	$\pm\alpha_4$	$\pm\alpha_2$

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