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ON APPROXIMATE SOLUTION OF TIME – FRACTIONAL ADVECTION-DISPERSION EQUATION

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Abstract- ***In this Paper, Modified differential transform method successfully applied for finding the approximate solution of the Time-Fractional Advection –Dispersion Equation. The fractional derivatives are considered in the Caputo sense. We discussed some numerical examples to demonstrate the efficiency and the accuracy of the proposed Method.***

Keywords: Caputo time-fractional derivative, Modified differential transform method (MDTM), Riemann–Liouville fractional integral operator, Time- Fractional Advection –Dispersion Equation

I. INTRODUCTION

Nowadays fractional differential equations (FDEs) or fractional Partial differential equations (FPDEs) have been used to Model a variety of problems in the field of mechanical engineering, physics, control theory, fluid mechanics, signal processing, viscoelasticity, electromagnetism,electrochemistry, thermal engineering, and many other physical processes [19],[14],[7],[12],[24]. Due to its Importance in several disciplines, many authors have been interested in studying the fractional calculus and finding accurate and efficient methods for solving FDEs or FPDEs. Some of the recent analytic/numerical methods are Adomian decomposition methods (ADM) [15],[18],[20], finite difference method [13], variational iteration method (VIM) [25],[5], fractional differential transform method [1], generalized differential transform method [17], operational matrix method [22], finite element method [23], Bernstein polynomial [21], iterative method [6] and the references therein.

In this paper, we consider [8] the Time- Fractional Advection –Dispersion Equation with the initial condition:

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=k(x,t)\frac{\partial^{2}u(x,t)}{\partial t^{2}}+v(x,t)\frac{\partial\, u(x,t)}{\partial t}+g(x,t),$$

x>0, t>0, 0<α≤1, (1)

u(x,0) = u_0(x),

(2)

where u(*x*, *t*), κ(*x*, *t*), and v (*x*, *t*) represent the solute concentration, the dispersion coefficient, and the average fluid velocity, respectively.
Fractional advection-dispersion equations are used in groundwater hydrology to model the transport of passive tracers carried by fluid flow in a porous medium.
In this paper, we apply Modified differential transform method (MDTM) [3], [2] to solve Time-Fractional Advection–Dispersion Equation. The concept of differential transform (one-dimension) was first proposed and applied to solve linear and nonlinear initial value problems in electric circuit analysis by Zhou [27].

Basic Definitions of Fractional Calculus:

Definition 1: A real function f(x), x>0, in the space c_μ, $\mu\in$ R if there exists a real number p> μ, such that f(x) = $x^p f_1(x)$, where $f_1(x)\in c[0,\infty)$ and it is said to be in the space c_μ^m if $f^m \in c_\mu$, m$\in$N.

Definition 2: The left-sided Riemann–Liouville fractional integral operator of order α ≥ 0, of a function f ∈ c_μ, μ≥-1 is defined as

$$I^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_{0}^{x}(x-t)^{\alpha-1}f(t)dt, \qquad \alpha>0, \qquad x>0$$

(3)
And J^0f(x) = f(x).

Definition 3: The fractional derivative of f(x) in the Caputo sense is defined as

$$D_{*}^{\alpha}f(x)=\frac{1}{\Gamma(m-\alpha)}\int_{0}^{x}(x-t)^{m-\alpha-1}f^{m}(t)dt$$

(4)

For m-1<α<m, m$\in N$, x>0, f ∈ c_{-1}^{n}. The unknown function f = f(x, t) is assumed to be a casual function of fractional derivatives (i.e., vanishing for α < 0) taken in Caputo sense as follows.

Definition 4: For m as the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order α > 0 is defined as

$$D_{*t}^{\alpha} f(x,t) = \frac{\partial^{\alpha} f(x,t)}{\partial t^{\alpha}}$$

$$= \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m f(x,t)}{\partial \tau^m}, \quad m-1 < \alpha < m$$

$$= \frac{\partial^m f(x,t)}{\partial t^m}, \alpha = m \in N \qquad (5)$$

Two-Dimensional Differential Transform Method:

Consider two variable function u(x, t) and suppose that u(x,t) can be represented as a product of two single-variable functions i.e., u(x, t) = f(x)g(t). Based on the properties of two-dimensional differential transform, the function u(x, t) can be represented as

$$u(x,t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U_{\alpha,1}(k,h)(x-x_0)^k (t-t_0)^{h\alpha} \qquad (6)$$

where $0 < \alpha$, $U_{\alpha,1}(k, h)$ is called the spectrum of u(x, t).

The generalized two-dimensional differential transform of the function u(x, t) is given by

$$U_{\alpha,1}(k,h) = \frac{1}{\Gamma(k+1)\Gamma(\alpha h+1)} \left[(D_{*x_0})^k (D_{*t_0}^{\alpha})^h u(x,t) \right]_{x_0,t_0} \qquad (7)$$

Where $(D_{*t_0}^{\alpha})^h = D_{*t_0}^{\alpha} \; D_{*t_0}^{\alpha} \; D_{*t_0}^{\alpha} \ldots\ldots\ldots\ldots \; D_{*t_0}^{\alpha}$ (h times). The function u(x, t) is represented by a finite series of (6) can be written as

$$u(x,t) = \sum_{k=0}^{l} \sum_{h=0}^{n} U_{\alpha,1}(k,h) x^k t^{h\alpha} + R_{\ln}(x,t) \; \ldots. \; (8)$$

and (6) implies that $R_{\ln}(x,t) = \sum_{k=l+1}^{\infty} \sum_{h=n+1}^{\infty} U_{\alpha,1}(k,h) x^k t^{h\alpha}$ is negligibly small. Usually, the values of l and n are decided by convergence of the series solution. In case of α = 1, the generalized two-dimensional differential transform (6) reduces to the classical two-dimensional differential transform [4], [7], [9], [26], [27].

Modified Differential Transform Method:

Even though, DTM is a convenient tool in the field of numerical approximations, it also encounters some complexity to determine the recursive relation for the problem containing nonlinear function. For instance, Let us consider the differential transform for

$$u^3(x,t) = \sum_{r=0}^{k} \sum_{q=0}^{k-r} \sum_{s=0}^{h} \sum_{p=0}^{h-s} U_{\alpha,1}(r,h-s-p) U_{\alpha,1}(q,s) U_{\alpha,1}(k-r-q,p)$$

(9)

(9) Involves four summations. Thus it is necessary to have a lot of computational work to calculate such differential transform $U_{1,\alpha}(h,k)$ for the large number of (k, h). Since, DTM is based on the Taylor series for all variables. To reduce the complexity in DTM, K. Aruna and A. Kanth in [3], [2] introduce the DTM with respect to the specific variable for the function u(x, t). Assume that the specific variable is the variable t then, we have the Taylor series expansion of the function u(x, t) at $t = t_0$ as follows.

$$u(x,t) = \sum_{h=0}^{\infty} \frac{1}{\Gamma(\alpha h+1)} \left(\frac{\partial^{\alpha h} u(x,t)}{\partial t^{\alpha h}} \right)_{t=t_0} (t-t_0)^{\alpha h}$$

(10)

Definition 5: The modified differential transform $U_{\alpha,1}(x, h)$ of u(x, t) with respect to the variable t at t_0 is defined by

$$U_{\alpha,1}(x,h) = \frac{1}{\Gamma(\alpha h+1)} \left(\frac{\partial^{\alpha h} u(x,t)}{\partial t^{\alpha h}} \right)_{t=t_0}$$

(11)

Definition 6: The modified differential inverse transform $U_{\alpha,1}(x, h)$ with respect to the variable t at t_0 is defined by

$$u(x,t) = \sum_{h=0}^{\infty} U_{\alpha,1}(x,h)(t-t_0)^{\alpha h}$$

(12) The fundamental mathematical operations performed by Modified differential transform method are listed in following Table .

Original function	Transformed function
$w(x,t) = u(x,t) \pm$	$W_{\alpha,1}(x,h) = U_{\alpha,1}(x,h) \pm V_{\alpha,1}(x,h)$
$w(x,t) = \mu u(x,t)$	$W_{\alpha,1}(x,h) = \mu U_{\alpha,1}(x,h)$
$w(x,t) = D_{*t}^{\alpha} u(x,t)$ 0<α≤1	$W_{\alpha,1}(x,t) = \frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} U_{\alpha,1}(x,h+$
$w(x,t) = (x-x_0)^m$	$W_{\alpha,1}(x,h) = (x-x_0)^m \delta(hx-n)$
$w(x,t) = u^2(x,t)$	$W_{\alpha,1}(x,h) = \sum_{m=0}^{h} U_{\alpha,1}(x,m) U_{\alpha,1}(x,h$
$w(x,t) = u^3(x,t)$	$W_{\alpha,1}(x,h) = \sum_{m=0}^{h} \sum_{l=0}^{m} U_{\alpha,1}(x,h-m) U_{\alpha,}$
$w(x,t) = \frac{\partial u(x,t)}{\partial t}$	$W_{\alpha,1}(x,h) = \frac{\partial U_{\alpha,1}(x,h)}{\partial x}$

Approximate Solutions of Time FADEs: In this section, we obtained solution of two Time- Fractional Advection – Dispersion Equation.

Example 1: Firstly, the following time FADE subject to the initial condition is considered [8]

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\mu\frac{\partial^{2}u(x,t)}{\partial t^{2}}-\frac{\partial u(x,t)}{\partial t},\ t>0,\ 0<\alpha\le 1, \tag{13}$$

$$u(x,0)=e^{-x} \tag{14}$$

The transformed version of (13) w. r. t. t is

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}U_{\alpha,1}(x,h+1)=\mu\frac{\partial^{2}U_{\alpha,1}(x,h)}{\partial x^{2}}-\frac{\partial U_{\alpha,1}(x,h)}{\partial x} \tag{15}$$

The transformed version of (14) is

$$U_{\alpha,1}(x,0)=e^{-x} \tag{16}$$

From the MDTM recurrence equation (15) we get $U_{\alpha,1}(x,h)$ values

$$U_{\alpha,1(x,1)}=\frac{(\mu+1)}{\Gamma(\alpha+1)}e^{-x},U_{\alpha,1}(x,2)=\frac{(\mu+1)^{2}}{\Gamma(2\alpha+1)}e^{-x},\ U_{\alpha,1}(x,3)=\frac{(\mu+1)^{3}}{\Gamma(3\alpha+1)}e^{-x}\ \ldots\ldots\ \ldots\text{and so on.}$$

Substituting $U_{\alpha,1}$'s into (12). We obtained the solution in the following form

$$u(x,t)=e^{-x}\left[1+\frac{(\mu+1)}{\Gamma(\alpha+1)}t^{\alpha}+\frac{(\mu+1)^{2}}{\Gamma(2\alpha+1)}t^{2\alpha}+\frac{(\mu+1)^{3}}{\Gamma(3\alpha+1)}t^{3\alpha}+\ldots\ldots\right] \tag{17}$$

In the limit of infinitely many terms, Eqs. (17) yields the solution.

$$u(x,t)=e^{-x}E_{\alpha}((1+\mu)t^{\alpha}) \tag{18}$$

where $E_{\alpha}((1+\mu)t^{\alpha})$ is the Mittag-Leffler function.

Which is same solution as the given in [8] Using FVIM.

Example 2: Now, the following time FADE subject to the initial condition is considered [8], [16].

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=k\frac{\partial^{2}u(x,t)}{\partial x^{2}}-v\frac{\partial u(x,t)}{\partial x},\quad t>0,\ 0<\alpha\le 1 \tag{19}$$

$$u\,(x,0)=\sin(x) \tag{20}$$

The transformed version of (19) w. r. t. t is

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}U_{\alpha,1}(x,h+1)=k\frac{\partial^{2}U_{\alpha,1}(x,h)}{\partial x^{2}}-v\frac{\partial U_{\alpha,1}(x,h)}{\partial x} \tag{21}$$

The transformed version of (20) is

$$U_{\alpha,1}(x,0)=\sin(x) \tag{22}$$

From the MDTM recurrence equation (21) we get $U_{\alpha,1}(x,h)$ values

$$U_{\alpha,1}(x,1)=\frac{1}{\Gamma(\alpha+1)}\{-(k\sin(x)+v\cos(x))\},$$

$$U_{\alpha,1}(x,2)=\frac{1}{\Gamma(2\alpha+1)}[(k^{2}-v^{2})\sin(x)+2kv\cos(x)],$$

$$U_{\alpha,1}(x,3)=\frac{1}{\Gamma(3\alpha+1)}[(v^{3}-3k^{2}v)\cos(x)+(3kv^{2}-k^{3})\sin(x)],\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Substituting $U_{\alpha,1}$'s into (12). We obtained the solution in the following form

$$u(x,t)=\sin(x)+\frac{t^{\alpha}}{\Gamma(\alpha+1)}\{-(k\sin(x)+\nu\cos(x))\}+$$
$$\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}[(k^2-\nu^2)\sin(x)+2k\nu\cos(x)]\ +$$
$$\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}[(\nu^3-3k^2\nu)\cos(x)+(3k\nu^2-k^3)\sin(x)]+..........\quad (23)$$

Which is same solution as the given in [8] using FVIM, [16] using ADM

CONCLUSIONS

In this paper, MDTM has been successfully applied to solve Time- Fractional Advection–Dispersion Equation. This method can obtain simple recursive equation. Compared with the ADM, FVIM , these illustrative problems shows that, MDTM does not required to find Adomian Polynomial like ADM and Lagrange multiplier like FVIM. Thus it is conclude that MDTM is very effective tool and it might be applicable for wide class of nonlinear fractional models in mathematical physics with high accuracy.

References:

1. A. Arikoglu and I. Ozkol, "Solution of fractional differential equations by using differential transform method," *Chaos, Solitons & Fractals*, vol. 34, no. 5, pp. 1473–1481, 2007.
2. K. Aruna and A. Kanth, "Approximate Solutions of Non-linear Fractional Schrodinger Equation Via Differential Transform Method and Modified Differential Transform Method," *Natl. Acad. Sci. Lett.*, vol. 32, no. 2, pp. 201–213, 2013.
3. K. Aruna and A. Kanth, "Two-Dimensional Differential Transform Method and Modified Differential Transform Method for Solving Nonlinear Fractional Klein–Gordon Equation," *Natl. Acad. Sci. Lett.*, vol. 37, no. 2, pp. 163–171, 2014.
4. C. Chen and S. Ho, "Solving partial differential equations by two-dimensional differential transform method," *Appl. Math. Comput.*, vol. 106, no. 2–3, pp. 171–179, 1999.
5. S. Das, "Analytical solution of a fractional diffusion equation by variational iteration method," *Comput. Math. with Appl.*, vol. 57, no. 3, pp. 483–487, 2009.
6. C. D. Dhaigude and V. R. Nikam, "Solution of fractional partial differential equations using iterative method," *Fract. Calc. Appl. Anal.*, vol. 15, no. 4, pp. 684–699, 2012.
7. A. E. Ebaid, "A reliable aftertreatment for improving the differential transformation method and its application to nonlinear oscillators with fractional nonlinearities," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 16, no. 1, pp. 528–536, 2011.
8. B. Ibiş and M. Bayram, "Approximate solution of time-fractional advection-dispersion equation via fractional variational iteration method.," *ScientificWorldJournal.*, vol. 2014, no. 0, p. 769713, 2014.
9. M.-J. Jang, C.-L. Chen, and Y.-C. Liu, "Two-dimensional differential transform for partial differential equations," *Appl. Math. Comput.*, vol. 121, no. 2–3, pp. 261–270, Jun. 2001.
10. F. Kangalgil and F. Ayaz, "Solitary wave solutions for the KdV and mKdV equations by differential transform method," *Chaos, Solitons & Fractals*, vol. 41, no. 1, pp. 464–472, 2009.
11. A. Kanth and K. Aruna, "Differential transform method for solving the linear and nonlinear Klein–Gordon equation," *Comput. Phys. Commun.*, vol. 180, no. 5, pp. 708–711, 2009.
12. A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, vol. 204. Elsevier, 2006.
13. M. M. Meerschaert and C. Tadjeran, "Finite difference approximations for two-sided space-fractional partial differential equations," *Appl. Numer. Math.*, vol. 56, no. 1, pp. 80–90, 2006.
14. K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley, 1993.
15. S. Momani and Z. Odibat, "Numerical approach to differential equations of fractional order," *J. Comput. Appl. Math.*, vol. 207, no. 1, pp. 96–110, 2007.
16. S. Momani and Z. Odibat, "Numerical solutions of the space□time fractional advection□dispersion equation," *Numer. Methods Partial Differ. …*, vol. 24, no. 6, pp. 1416–1429, 2008.
17. Z. Odibat, S. Momani, and V. Erturk, "Generalized differential transform method: application to differential equations of fractional order," *Appl. Math. Comput.*, vol. 197, no. 2, pp. 467–477, 2008.
18. Z. Odibat and S. Momani, "Numerical methods for nonlinear partial differential equations of fractional order," *Appl. Math. Model.*, vol. 32, no. 1, pp. 28–39, 2008.
19. I. Podlubny, "Fractional differential equations, Acad," *Press. London*, 1999.
20. S. S. Ray and R. K. Bera, "Analytical solution of the Bagley Torvik equation by Adomian decomposition method," *Appl. Math. Comput.*, vol. 168, pp. 398–410, 2005.
21. D. Rostamy and K. Karimi, "Bernstein polynomials for solving fractional heat-and wave-like equations," *Fract. Calc. Appl. Anal.*, vol. 15, no. 4, pp. 556–571, 2012.
22. A. Saadatmandi and M. Dehghan, "A new operational matrix for solving fractional-order differential equations," *Comput. Math. with Appl.*, vol. 59, no. 3, pp. 1326–1336, 2010.
23. J. Sabatier, O. P. Agrawal, and J. a. T. Machado, *Advances in Fractional Calculus*. 2007.
24. S. Samko, A. A. Kilbas, and O. Marichev, *Fractional Integrals and Derivatives*. Taylor & Francis, 1993.
25. [25] N. H. Sweilam, M. M. Khader, and R. F. Al-Bar, "Numerical studies for a multi-order fractional differential equation," *Phys. Lett. A*, vol. 371, pp. 26–33, 2007.
26. [26] a. Tari and S. Shahmorad, "Differential transform method for the system of two-dimensional nonlinear Volterra integro-differential equations," *Comput. Math. with Appl.*, vol. 61, no. 9, pp. 2621–2629, 2011.
27. [27] J. Zhou, "Differential Transformation and Its Applications for Electronic Circuits, Huazhong Science & Technology University Press, China," 1986.