FISHER- KOLMOGOROFF AND RELATED REACTION DIFFUSION EQUATIONS

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Abstract- *In this paper, we study the travelling wave solution of the Fisher-Kolmogoroff equation. We also study another related Reaction Diffusion equation. We analyze the stability of these two systems through phase plane analysis. We perform some simulations.*
Keywords: Fisher- Kolmogoroff Equation, Reaction diffusion equation, Equilibrium points, Energy level lines.

I. INTRODUCTION

A reaction-diffusion equation comprises of a reaction term and a diffusion term, a typical Reaction Diffusion equation is as follows:

$$\frac{\partial u}{\partial t} = D \Delta u + f(u)$$

$u = u(x,t)$ is a variable which describes concentration of a substance or the population at time *t*. Δ denotes the Laplace operator. So the first term on the right hand side describes the "diffusion", including D as diffusion coefficient. The second term, $f(u)$ is a smooth function $f : \mathbb{R} \rightarrow \mathbb{R}$ representing reaction kinetics.

Instead of a scalar equation, one can also introduce systems of reaction diffusion equations, which are of the form

$$\frac{\partial u}{\partial t} = \nabla_x \left(diag(d_1, d_2, \ldots d_m) \right) \nabla_x u + f(t, x, u, \nabla_x)$$

(1) where $t \geq 0$ denotes time and $x \in \Omega \subset \mathbb{R}^d$ denotes position within a d- dimensional bounded domain Ω with a smooth boundary. The States

$u = (u_1, u_2, \ldots, u_m), i = 1,2,\ldots, m$

describe the concentrations or densities of substances or populations, $u_i = u_i(t,x)$. The functions $d_i(t,x)$ are called diffusion coefficients or diffusivities. On Ω, the bounded domain introduced earlier, boundary conditions need to be specified for equations (1). Typical

conditions would include either Dirichlet, which prescribes the value at the boundary $u_i(x,t) = f_D(t,x), x \in \partial\Omega$ or Neumann conditions giving the diffusion flux through the boundary $\partial\Omega\, d_i(t,x)\nabla_x u_i(t,x) = f_N(t,x)$ or mixed boundary conditions.

One of the fascinating aspects of the natural world is the diversity of shapes that make up the animal and plant kingdoms. How these patterns arise is one of the mysteries of science. The problem that Turing (Turing, 1952) addressed in his seminal paper, "The chemical basis of morphogenesis"

was precisely this. He presented a theory in which he proposed that cells actually respond to a chemical pre-pattern.

He considered a system of morphogens reacting and diffusing in such a way that, in the absence of diffusion, they exhibited

a spatially uniform steady state which would be stable. This phenomenon, termed diffusion-driven instability, has now been shown to occur in chemistry. Experimental results illustrate the formation of striped and spotted patterns, as well as more complicated patterns. This was the first example of what is now called an emergent phenomenon in the sense that the behavior of the system, in this case a patterning instability, emerges from the components and is not part of the components. In his system, the reaction kinetics are stabilizing and we know that diffusion is stabilising in the sense that it homogenizes spatial patterns. Therefore, two stabilizing systems interacted to produce an instability. In other words, he recognized that it was the integration of components that gave rise to the structures and behaviours we observe, rather than each behaviour being encoded in its own component. Many of these patterns can be exhibited by Turing models and there is now a vast amount of theoretical and experimental literature in this area (see (Murray, 2002) , for a review). Turing systems have the form

$$\frac{\partial U}{\partial t} = D_u \nabla^2 U + f(U,V)$$
$$\frac{\partial V}{\partial t} = D_v \nabla^2 V + g(U,V) \tag{2}$$

These equations describe the evolution of the concentrations, $U(\bar{x},t)$, $V(\bar{x},t)$ at spatial position $\bar{x}$ and time *t*, of two chemicals due to diffusion, with constant diffusion coefficients D_u , D_v respectively, and reaction, modeled by the functions, *f* and g which are typically non-linear. We now examine solutions of the travelling-wave type for reaction-diffusion equations.

2. Travelling Wave Solution

Our aim is to examine solutions of the form (P. K. Maini, 1997)

$$u(x,t) = \varphi(x - ct) \text{ for } (2).$$

A travelling wave solution is a wave which travels without change of shape. If a solution $u(x,t)$ represents a travelling wave the shape of the solution remains the same for all t, in addition to this the speed of the propagation of this wave is a constant, denoted by c, symbolically we represent this as $u(x,t) = u(x-ct) = U(z)$,where $z = x-ct$

(3)   Then $u(x,t)$ is a travelling wave which moves with a constant speed c in the positive direction of the X axis. Symmetrically we can make the statement that a wave which moves in the negative X direction has the form $u(x+ct)$, z is called the the wave variable. This kind of a solution is a means of separating the variables in a partial differential equation. Let us observe that in a traveling wave solution or solutions in x and t in the form (3) gives

$$\frac{\partial u}{\partial t} = -c\frac{\partial U}{\partial z} \quad \text{and} \quad \frac{\partial u}{\partial x} = \frac{\partial U}{\partial z}.$$

Thus partial differential equations in x and t are converted to the ordinary differential equations in new variable z. Here $U(z)$ is bounded for all z and nonnegative. The reason being that the entities with which we deal with are chemicals or populations.

3. Fisher–Kolmogoroff Equation:

A reaction diffusion equation is of the form

$$\frac{\partial u}{\partial t} = f(u) + D\nabla^2 u \quad \text{where} \quad f(u) \quad \text{is a nonlinear function}$$

with $f(u) = ku(1-u)$, where k and D are positive parameters. It was suggested by Fisher (Fisher, 1937) as a deterministic version of a stochastic model for the spatial spread of a favoured gene in a population. It is also the natural extension of the logistic growth population model when the population disperses via linear diffusion. This equation and its travelling wave solutions have been widely studied, as has been the more general form with an appropriate class of functions $f(u)$ replacing $ku(1-u)$. We discuss this model equation in some detail, not because in itself it has such wide applicability but because it is the prototype equation which admits travelling wavefront solutions.

In a collection of particles like cells, bacteria, chemicals, animals each particle move around in a random way. When this microscopic irregular movement results in gross regular motion of the group, we think of it as a diffusion process. The simplest nonlinear reaction diffusion equation is

$$\frac{\partial u}{\partial t} = ku(1-u) + D\frac{\partial^2 u}{\partial x^2}$$

$$(4)$$

, (Fisher, 1937) where k and D are positive parameters.

Let $t^* = k\,t$, $x^* = x\left(\frac{k}{D}\right)^{1/2}$

$$\frac{\partial t}{\partial u} = \frac{1}{k}\frac{\partial t^*}{\partial u} \quad , \quad \frac{\partial x}{\partial u} = \left(\frac{D}{k}\right)^{1/2}\frac{\partial x^*}{\partial u}$$

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{k}{D}\right)\frac{\partial^2 u}{\partial x^{*2}}$$

Equation (4) reduces to $k\frac{\partial u}{\partial t^*} = k\,u\,(1-u) + D\frac{k}{D}\frac{\partial^2 u}{\partial x^{*2}}$

Omitting asterisks,

$$\frac{\partial u}{\partial t} = u(1-u) + \frac{\partial^2 u}{\partial x^2}$$

(5) The steady states are $u=0$ *and* $u=1$ which are unstable and stable.

We look for solutions of (2) for which $0 \le u \le 1$.

Travelling wave solution is of the form $u(x,t) = U(z)$, $z = x-ct$ (6)

where c is wave speed. Equation (4) is invariant for x. c may be positive or negative. We assume $c \ge 0$. From (6),

$$\frac{\partial u}{\partial t} = -c\frac{\partial U}{\partial z} \quad , \quad \frac{\partial u}{\partial x} = \frac{\partial U}{\partial z} \quad , \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 U}{\partial z^2}$$

(5) reduces to

$$-c\frac{\partial U}{\partial z} = U(1-U) + \frac{\partial^2 U}{\partial z^2}$$

$$U'' + cU' + U(1-U) = 0$$

(7) We have to find values of c, for which the wave form solution U satisfies $\lim_{z\to\infty} U(z) = 0$, $\lim_{z\to -\infty} U(z) = 1$

Let $U' = V$ From (7), $U'' = V' = -cV - U(1-U)$

(8)This implies $\dfrac{dV}{dU} = \dfrac{V}{-cV - U(1-U)} = \dfrac{f(U,V)}{g(U,V)}$

In order to obtain equilibrium points, we choose

$f = V = 0$

$g = -cV - U(1-U) = 0$

Let $(\bar{U}, \bar{V})$ be an equilibrium point. At equilibrium point, $f(\bar{U},\bar{V}) = g(\bar{U},\bar{V}) = 0$.

The two equilibrium points $(\bar{U}, \bar{V})$ are (0, 0) and (1, 0).

The Jacobian matrix J is

$$J = \begin{bmatrix} f_U & f_V \\ g_U & g_V \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1+2U & -c \end{bmatrix}$$

At the equilibrium point (0,0), the Jacobean matrix is J

$$= \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix}$$

The Eigenvalues of J are the roots of the characteristic equation

$$\lambda^2 + c\lambda + 1 = 0$$

The two roots of this characteristic equation are

$$\lambda_1 = \frac{-c+\sqrt{c^2-4}}{2} \quad \text{and} \quad \lambda_2 = \frac{-c-\sqrt{c^2-4}}{2}$$

Looking at Eigenvalues , it can be seen that if $c^2 > 4$ the solution is a stable node and if $c^2 < 4$ (the Eigenvalues become complex) the solution is a stable spiral.

At the equilibrium point (1, 0), the Jacobian matrix is J =

$$\begin{bmatrix} 0 & 1 \\ 1 & -c \end{bmatrix}$$

The characteristic equation is $\lambda^2 + c\lambda - 1 = 0$. The two roots of this characteristic equation are $\lambda_1 = \frac{-c+\sqrt{c^2+4}}{2}$ and $\lambda_2 = \frac{-c-\sqrt{c^2+4}}{2}$. These roots are real and unequal. The equilibrium point (1, 0) is a saddle point.

The phase plane trajectories in the figure show the stable and unstable solution.

Let c_{min} be the minimum value of c where c defined in equation (3) is the wave speed. If $c \ge c_{min} = 2$ the origin which is one of the equilibrium points is a stable node. For $c^2 < 4$,as mentioned earlier the origin is a stable spiral.

Take $c = 3 > 2$. Equations (8) become $U' = V$

$$U'' = V' = -3V - U(1-U)$$

The phase portraits using Mathematica are as follows :

fig.1 represents the solution of equations (9) with initial conditions U(0)=0,V(0)=1

fig.2 represents the solution of equations (9) with initial conditions U(0)=0,V(0)= -1

fig.3 represents the solution of equations (9) with initial conditions U(0)=0, and different values of V(0)=1, 0.8, 0.6, 0.4, 0.2, -0.2, -0.4, -0.6, -0.8, -1.

Fig:1   Fig:2   Fig:3
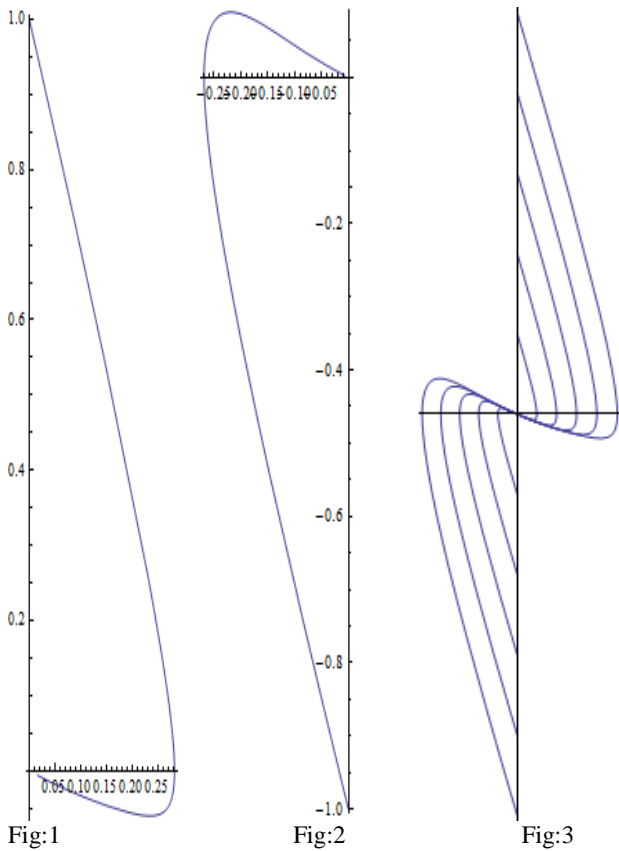
4. A Particular Form of the Fisher Equation:

If the coefficient k in equation (4) is of the form $k = k(x)$ then the Fisher equation becomes

$$\frac{\partial u}{\partial t} = k(x) U (1 - U) + D \bar{\nabla}^2 U$$

Using the Travelling wave solution method stated in the previous section this equation becomes (after choosing the coefficient D=1) $U'' + k(x) U (1 - U) = 0$, where the derivative is w.r.t. z, $z = x - ct$.

This gives the system of equations
$$V' = -k(x) U (1 - U)$$

(9) Let us choose $k(x) = sgn(x) = \begin{cases} -1, & if\ x < 0 \\ 0, & if\ x = 0 \\ 1, & if\ x > 0 \end{cases}$

(10) The equilibrium points of (9) are given by
$U' = 0$, $V' = 0$.
The equilibrium points are $(0,0)$ and $(1,0)$.

The Jacobian matrix is $J = \begin{bmatrix} U'_U & U'_V \\ V'_U & V'_V \end{bmatrix}$

$J = \begin{bmatrix} 0 & 1 \\ k(x)(2u - 1) & 0 \end{bmatrix}$ $J_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -k(x) & 0 \end{bmatrix}$

and $J_{(1,0)} = \begin{bmatrix} 0 & 1 \\ k(x) & 0 \end{bmatrix}$

I. For the equilibrium point $(0,0)$:

The characteristic equation is $\lambda^2 - (0)\lambda + k(x) = 0$
The Eigenvalues are

$\lambda_1 = \sqrt{-k(x)}$ and $\lambda_2 = -\sqrt{-k(x)}$

Case(i): When $x < 0$, $\lambda_1 = 1, \lambda_2 = -1$.
The roots are real with opposite signs. The point $(0,0)$ is a saddle point
Case(ii): When $x = 0$, $\lambda_1 = 0, \lambda_2 = 0$. The roots are real and equal. The point $(0,0)$ is a node.
Case(iii): When $x > 0$, $\lambda_1 = i, \lambda_2 = -i$. The roots are purely imaginary. The point $(0,0)$ is a center.

II. For the equilibrium point $(1,0)$:

The characteristic equation is
$$\lambda^2 - (0)\lambda - k(x) = 0$$
$$\lambda^2 = k(x)$$

The Eigenvalues are $\lambda_1 = \sqrt{k(x)}$ and

$\lambda_2 = -\sqrt{k(x)}$
Case(i): When $x < 0$, $k(x) = -1$, $\lambda_1 = i$, $\lambda_2 = -i$. The point $(1,0)$ is a center.
Case(ii): When $x = 0$, $k(x) = 0$, . . $\lambda_1 = 0, \lambda_2 = 0$ .The point (1, 0) is a node.
Case(iii): When $x > 0$, $k(x) = 1$, . $\lambda_1 = 1, \lambda_2 = -1$ .The point (1,0) is a saddle point.

We now calculate the energy level lines of the system of equations (9).

Set $k(x) = k$, where k takes on values 1,-1 and 0.

From (9), $\frac{dV}{dU} = \frac{-k U (1-U)}{V}$

$V\, dV = -k\, U\, (1 - U) dU$

Suppose $U = U_0$, $V = V_0$, at $t = t_0$.

$$\int_{V_0}^{V} V\, dV = -k \int_{U_0}^{U} U\,(1 - U)\, dU$$

$$\frac{1}{2} V^2 + k \int_{U_0}^{U} U\,(1 - U)\, dU = \frac{1}{2} V_0^2 + k \int_{U_0}^{U} U\,(1 - U)\, dU$$

$$\frac{1}{2} V^2 + k \left[\frac{U^2}{2} - \frac{U^3}{3} \right] = \frac{1}{2} V_0^2 + k \left[\frac{U_0^2}{2} - \frac{U_0^3}{3} \right]$$

$$= E(U, V) = h = \text{Total Energy.}$$

Case(i): When $x < 0$, $k = -1$, the Total Energy is

$$E(U, V) = h = \frac{1}{2} V^2 + \frac{U^3}{3} - \frac{U^2}{2}$$

To obtain the individual energy level lines we choose some particular points through which the level lines pass. We plot the energy level curves using Mathematica.
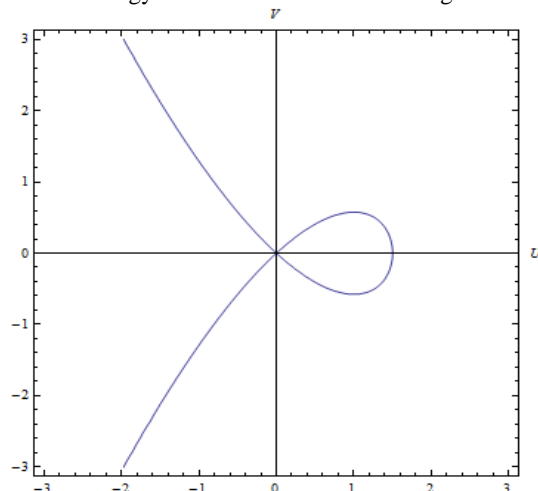

Figure 4: The energy level line for $k = -1$ passing through the point (0, 0) is

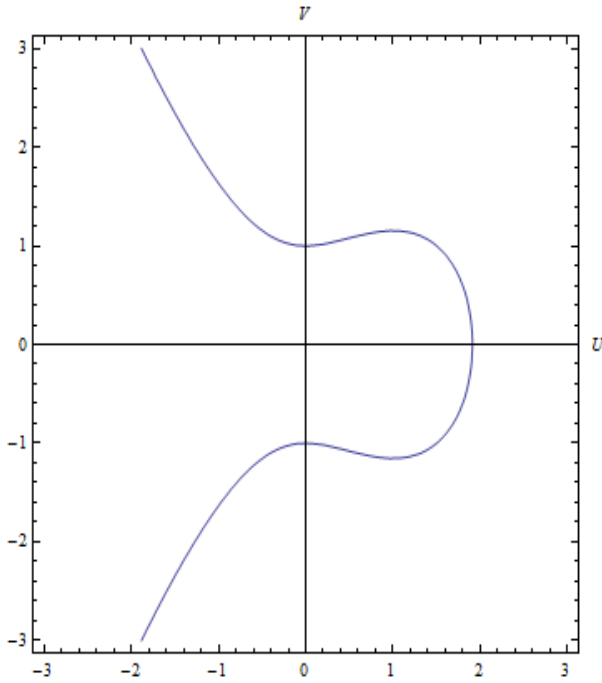$h = E(0,0) = \frac{1}{2}V^2 + \frac{1}{3}U^3 - \frac{1}{2}V^2 = 0$

Figure 5: The energy level line for $k = -1$ passing through the point (0, 1) is

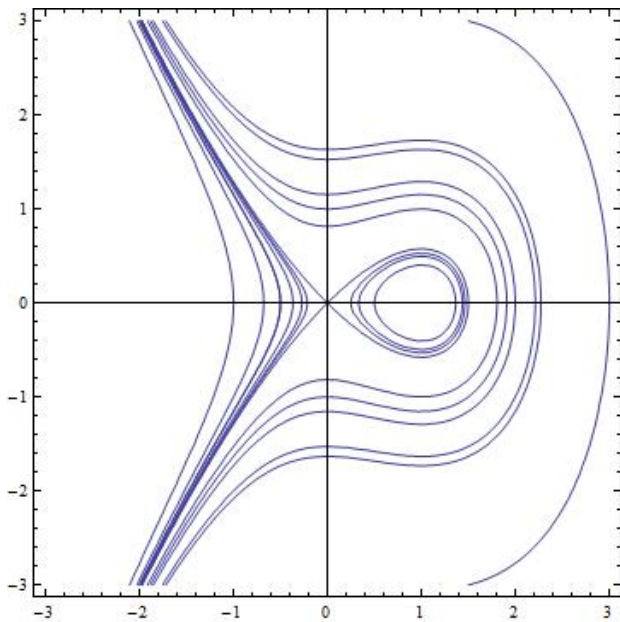$h = E(0,1) = \frac{1}{2}V^2 + \frac{1}{3}U^3 - \frac{1}{2}V^2 = \frac{1}{2}$

Figure 6: The energy level lines passing through different points

Case (ii) : When $k = 0$, The general energy equation is
$\frac{1}{2}V^2 + k\left[\frac{U^2}{2} - \frac{U^3}{3}\right] = E(U,V)$

The level line through any general point (a,b) is
$\frac{1}{2}b^2 + k[\frac{a^2}{2} - \frac{a^3}{3}] = E(a,b)$.

The equation of the level line through the point (a,b) is

$\frac{1}{2}V^2 + k\left[\frac{U^2}{2} - \frac{U^3}{3}\right] = \frac{1}{2}b^2 + k[\frac{a^2}{2} - \frac{a^3}{3}]$

When $k = 0$, it reduces to $V^2 = b^2$.
In this case energy level curves are straight lines of the form $V = constant$ which are parallel to the U axis.

Case (iii) : When $k(x) = 1$. $E(U,V) = \frac{1}{2}V^2 + \frac{1}{2}U^2 - \frac{1}{3}U^3$
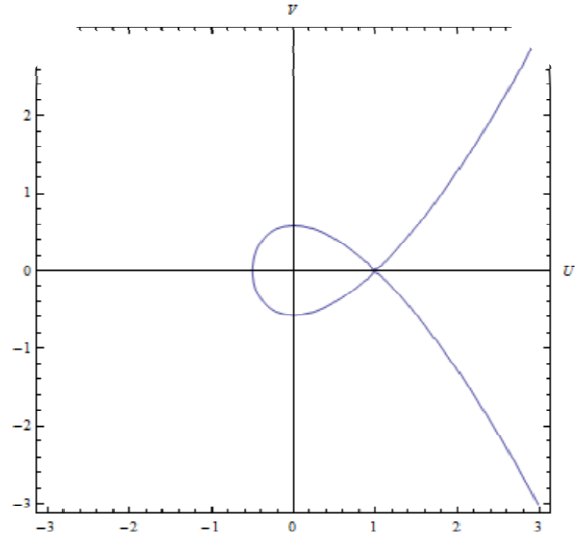
We plot energy level curves for different values of E as

Figure 7: The energy level line for $k = 1$ passing through the point (1, 0) is

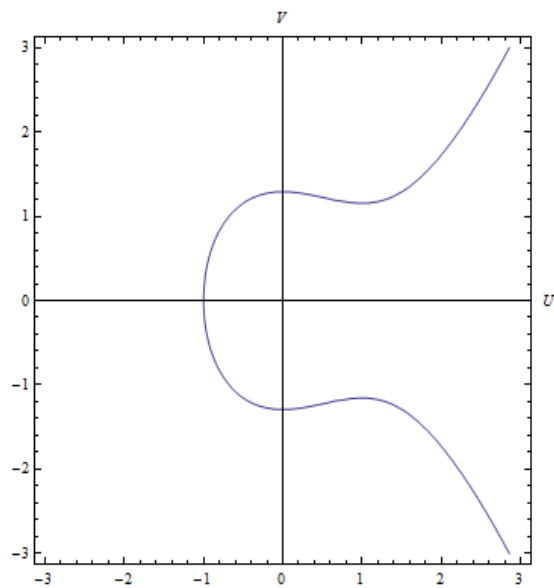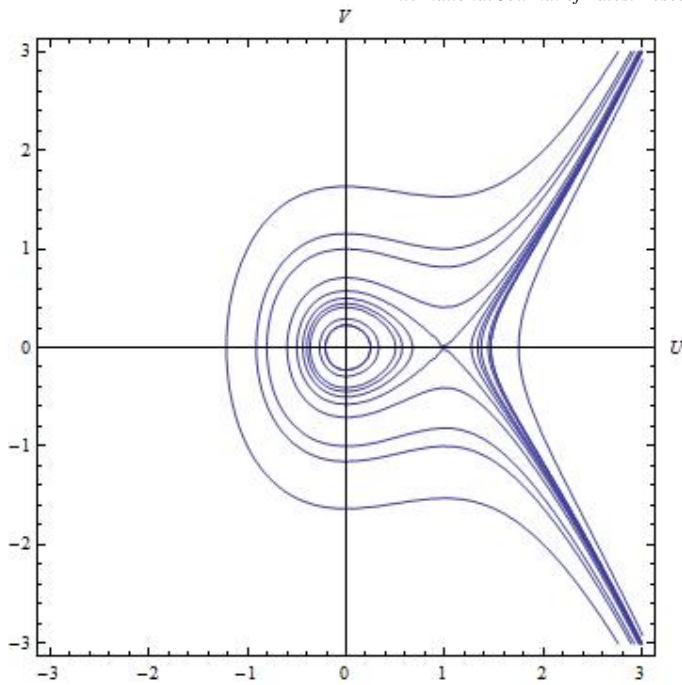$h = E(1,0) = \frac{1}{2}V^2 + \frac{1}{2}U^2 - \frac{1}{3}U^3 = \frac{1}{6}$.

Figure 8: The energy level line for $k = 1$ passing through the point (-1, 0) is

$h = E(-1,0) = \frac{1}{2}V^2 + \frac{1}{2}U^2 - \frac{1}{3}U^3 = \frac{5}{6}$.

Figure 9: The energy level lines passing through different points.

Conclusion:

We performed the phase plane analysis after using a travelling wave solution for the general Fisher-Kolmogoroff Equation and a particular form of the Fisher-Kolmogoroff Equation. For the general form, we obtained two equilibrium points and in particular form of the Fisher-Kolmogoroff Equation we obtained some closed orbit and a homoclinic orbit.

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