Abstract- For analytic functions \( f(z) \) normalized with \( f(0) = 0 \) and \( f'(0) = 1 \) in the open unit disk \( U \), a new class satisfying some conditions with some complex number \( \lambda \) and some real number \( \alpha \) is introduced. In the present paper necessary and sufficient condition for \( f(z) \) is obtained. Also some properties of the same class are obtained.

Keywords: Analytic and univalent functions; Ruscheweyh derivative; \( \alpha \) convex functions; Coefficients bounds; etc.

I. INTRODUCTION

Let \( \Delta \) be the class of analytic functions defined on the open unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) and normalized by the conditions \( f(0) = 0 \) and \( f'(0) = 1 \). A function \( f \in \Delta \) has Taylor's series expansion of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n.\tag{1.1}
\]

Let \( R(\alpha) \) denote the subclass of \( \Delta \) consisting of functions \( f(z) \) which satisfy

\[
\text{Re} f'(z) > \alpha (z \in \mathbb{D}) \quad \text{for some real } \alpha (0 \leq \alpha < 1).\tag{2.1}
\]

A function \( f(z) \in R(\alpha) \) is said to be close-to-convex of order \( \alpha \) in \( \mathbb{D} \) (cf. Goodman[1]). We know that \( R(\alpha_1) \subset R(\alpha_2) \) if \( 0 \leq \alpha_1 \leq \alpha_2 < 1 \) and \( R(\alpha) \subset \Delta \) by Noshiro-Warschawski theorem (cf. Duren[2]).

Given two functions \( f \), \( g \in \Delta \), where \( f(z) \) is given by (1.1) and \( g(z) \) is given by

\[
g(z) = z + \sum_{n=2}^{\infty} b_n z^n.
\]

The Hadamard product \( f \) and \( g \), denoted by \( f \ast g \), is defined by

\[
f \ast g(z) = z + \sum_{n=2}^{\infty} (a_n + b_n) z^n.
\]

The Ruscheweyh derivative of order \( k \) is denoted by

\[
D^k f(z) = z + \sum_{n=2}^{\infty} b_n^{(k)} z^n, \quad k > -1, \quad z \in \mathbb{D}
\]

where

\[
b_n^{(k)} = \frac{(n+k-1) \cdots (n+1)}{(n+k) \cdots (n+1)} a_n, \quad k = 0, 1, 2, \ldots
\]

and

\[
D^k f(z) = 1 + \sum_{n=2}^{\infty} b_n^{(k)} z^n.
\]

II. Properties of the class \( \Delta^{\ast}(\beta, \lambda) \):

Theorem 2.1: A function \( f(z) \) defined by (1.1) is in the class \( \Delta^{\ast}(\beta, \lambda) \) if

\[
\sum_{n=2}^{\infty} n b_n (1 + \lambda |\beta| |b_n|) a_n |z|^n \leq |z|^\lambda |\beta| |a_n| \quad (2.1)
\]

Proof: Since

\[
\frac{\sum_{n=2}^{\infty} n b_n |z|^n}{\sum_{n=2}^{\infty} (1 + \lambda |\beta| |b_n|) a_n |z|^n} \leq \frac{|z|^\lambda |\beta| |a_n|}{\sum_{n=2}^{\infty} n b_n |z|^n}
\]

Therefore, if \( f(z) \) satisfies the inequality (2.1), then

\[
\sum_{n=2}^{\infty} n b_n (1 + \lambda |\beta| |b_n|) a_n |z|^n \leq |z|^\lambda |\beta| |a_n|
\]

using this we have
Theorem 2.2: A function $f(z)$ defined by (1.1) is in the class $D^n_\kappa(\beta_1, \beta_2, \lambda)$ if and only if

$$\sum_{n=1}^{\infty} n^2 \beta_n (\lambda) \leq \lambda$$

(2.1)

Proof: In view of theorem (2.1), we need only to prove that necessity.

If $f(z) \in D^n_\kappa(\beta_1, \beta_2, \lambda)$ then,

$$\sum_{n=1}^{\infty} n^2 \beta_n (\lambda) \leq \lambda$$

Since $\sum_{n=1}^{\infty} n^2 \beta_n (\lambda) \leq \lambda$, we have

$$\sum_{n=1}^{\infty} n^2 \beta_n (\lambda) \leq \lambda$$

The above condition must hold for all values of $z; |z| < 1$.

Corollary 2.1: If $f(z) \in D^n_\kappa(\beta_1, \beta_2, \lambda)$

then $|a_n| \leq \frac{1}{n^2 \beta_n (\lambda)} (n = 2, 3, \ldots)$.

Proof: By theorem (2.1), if $f(z) \in D^n_\kappa(\beta_1, \beta_2, \lambda)$

then $\sum_{n=1}^{\infty} n^2 \beta_n (\lambda) \leq \lambda$.

Theorem 2.3: Let $f(z)$ defined by (1.1) and $g(z)$ defined by (1.2) be in the class $D^n_\kappa(\beta_1, \beta_2, \lambda)$. Then the function $h(z) = \xi f(z) + (1-\xi) g(z) = z + \sum_{n=1}^{\infty} \sigma_n z^n$, where $\sigma_n = \xi a_n + (1-\xi) b_n$,

(0 ≤ \xi ≤ 1) belongs to the class $D^n_\kappa(\beta_1, \beta_2, \lambda)$.

Proof: Since $f(z)$ and $g(z) \in D^n_\kappa(\beta_1, \beta_2, \lambda)$, we have

$$\sum_{n=1}^{\infty} n \beta_n (\lambda) \leq |a_n| \leq \frac{1}{n^2 \beta_n (\lambda)} (n = 2, 3, \ldots)$$

Clearly, $h(z) = \sum_{n=1}^{\infty} \sigma_n z^n$, where \( \sigma_n = \xi a_n + (1-\xi) b_n \),

Hence, $h(z) \in D^n_\kappa(\beta_1, \beta_2, \lambda)$.

References: