ON SINGLE INTEGRALS INVOLVING HYPERGEOMETRIC FUNCTIONS AND I-FUNCTION

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Abstract - This paper consists of single integrals involving hyper geometric function and I-function. Single integrals are obtained with the help of known results given by Devra [14] and the continuous function relations given in Rainville (1960). These single integrals involving hypergeometric function with I-function are obtained with the help of known integrals given by HTF Volume - I, McGrawhill, New York, [PP-115-116],(1953). Since I-function is one of the most generalized functions of one variable, So by specializing the parameters, we can obtain some new special cases, which are interesting and believed to be new, has also been given.  Keyword - H-Function, Continuous function, Single integrals, G-Function, Mellin-Barnes integral, Contour integrals, I-function, Watson’s Formulae.

1. INTRODUCTION

In 1812,C.F.Gauss introduced the series which represented by the symbols \( _2F_1 \) \(( \alpha, \beta, \gamma, z) \) and defined as

\[
_2F_1(\alpha, \beta, \gamma, z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{z^n}{n!}
\]

provided \( \Re(\gamma) > 0 \) \hspace{1cm} (1.1)

where \( \gamma \) is neither zero nor a negative integer.

A natural generalization of \( _2F_1 \) is the generalized hypergeometric function is denoted by \( _pF_q \) which is defined in the following

\[
_pF_q \left[ \begin{array}{c}
\alpha_1, \ldots, \alpha_p \\
\beta_1, \ldots, \beta_q
\end{array} ; z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{z^n}{n!}
\]

\hspace{1cm} (1.2)

In 1908, Barnes represented the Gauss hypergeometric function \( _2F_1 \) by a contour integral which after the work of Mellin (1910) become popular as Mellin-Barnes type contour integrals.

In 1992 and 1994 Lavoicetal, Grondin and Rathie [30] and [31] have obtained twenty five Summation formulæ closely related to Watson’s theorem. Single integral involving hypergeometric function with I function are also obtained with help of Watson’s Formulae. Watson’s Formulae are given below which are used to prove the result in this paper
Provided
Re (2c – a – b) > – 2


\[ (\theta + c - 1) _2F_1(a, b, c; z) = (c - 1) _2F_1(a, b; c; z) \]  

---(1.11)

Standard Known Integral

\[
\int_0^\frac{\pi}{2} e^{\theta(\cos \theta)^{\mu-1}(\sin \theta)^{\nu-1}} _2F_1(a, b; c; e^{\theta} \cos \theta) d\theta
\]

Provided
Re (\mu) > 0, Re (\sigma) > 0, Re (c + \sigma - a - b) > 0

---------------(1.12)

If we put

\[
\mu = c, \sigma = c + j \text{ and } c = \frac{1}{2} (a + b + i + 1), j = 0, i = 1
\]

Then

\[
\int_0^{\pi/2} e^{2\pi \theta (\cos \theta)^{\mu-1}(\sin \theta)^{\nu-1}} \left( a-b \right) _2F_1(a, b, c; \frac{a+b+2}{2} e^{\theta} \cos \theta) d\theta = 2^{a+b-2c} e^{\frac{\alpha \theta}{2}} \frac{a+b+1}{c} - \frac{a-b}{2}
\]

---(1.13)

Duplication formula

\[
2m = \sqrt{\frac{m+1}{2}} \sum_{j=m}^{2m-1}
\]

Provided by Re (c) > 0, Re (2c – a – b) > 0

---------------(1.14)

In this paper, we extend H-function to find the single integral. Section 2 and 3 propose the statement and proof of single integral with hyper geometric function and I-function respectively. Section 4 represents the special case. Finally section 5 represents the conclusion of this paper.

2. STATEMENT

\[
\int_0^{\pi/2} e^{2\pi \theta (\cos \theta)^{\mu-1}(\sin \theta)^{\nu-1}} \left( a-b \right) _2F_1(a, b, c; \frac{a+b+2}{2} e^{\theta} \cos \theta)
\]

\[
\times \left[ \begin{array}{c}
\frac{a + b + 1}{2} \\
\frac{c - a + 1}{2} \\
\frac{c - b}{2}
\end{array} \right]
\]

In this paper, we use the result of Rainville (pp-51)

\[
(a - b) _2F_1(a, b; c; z) = a _2F_1(a + 1, b; c; z) - b _2F_1(a, b + 1; c; z)
\]

----------(1.9)

\[
(\theta + a) _2F_1(a, b; c; z) = a _2F_1(a + 1, b; c; z)
\]

----------(1.10)
\[ \frac{2^{s+b-2c} e^{\frac{\pi}{2}\sqrt{a-b+1}}}{|a-b|} = \frac{1}{2\pi i} \int_{L} \theta(s)z^{1/2} e^{(2\alpha c+2\alpha s)\theta}(\cos\theta)^{c+\lambda s-1}(\sin\theta)^{c+\lambda s-1} \]

\[ \left\{ (a-b)_{R} \times \int_{L} e^{(2\alpha c+2\alpha s)\theta}(\cos\theta)^{c+\lambda s-1}(\sin\theta)^{c+\lambda s-1} \right\} d\theta \]

Evaluate the inner integral with the help of (1.13) and replace c \to c + \lambda s, we have

\[ \frac{1}{2\pi i} \int_{L} \theta(s)z^{1/2} e^{(2\alpha c+2\alpha s)\theta}(\cos\theta)^{c+\lambda s-1}(\sin\theta)^{c+\lambda s-1} \]

\[ \left\{ (a-b)_{R} \times \int_{L} e^{(2\alpha c+2\alpha s)\theta}(\cos\theta)^{c+\lambda s-1}(\sin\theta)^{c+\lambda s-1} \right\} d\theta \]

Provided by:- The I-function occurring on both sides as analytic and satisfied the conditions as given with the definition of I-function.

3. PROOF
In this section we give the proof of the section 2. The statement of section 2 is denoted by \( I \) and expressing the I-function by means of its contour integral as given in (1.8) we have

\[ I = \int_{0}^{\pi/2} e^{2\alpha c \theta}(\cos\theta)^{c+1}(\sin\theta)^{c+1} \left\{ (a-b)_{R} \right\} \left\{ 1+b+2 \right\} e^{i\theta(\cos\theta)} d\theta \]

\[ \left[ \frac{1}{2\pi i} \int_{L} \theta(s)z^{1/2} e^{2\alpha c \theta}(\cos\theta)^{c+1}(\sin\theta)^{c+1} ds \right] d\theta \]

where

\[ \theta(s) = \sqrt{-1} \text{ and } \theta(s) = \sum_{j=1}^{R} \prod_{j=1}^{n} (\beta_{j}-B_{j}) \prod_{j=1}^{n} (1-\alpha_{j}+A_{j}) \]

Changing the order of integration, which is easily seen to be justified by the application of De-lavallee Poussin's Theorem [2, p. 504], we have

\[ 2^{s+b-2c} e^{\frac{\pi}{2}\sqrt{a-b+1}} = \frac{1}{|a-b|} \int_{L} \theta(s)z^{1/2} e^{(2\alpha c+2\alpha s)\theta}(\cos\theta)^{c+\lambda s-1}(\sin\theta)^{c+\lambda s-1} \]

where

\[ K = \frac{1}{2\pi i} \int_{L} \theta(s)z^{1/2} e^{(2\alpha c+2\alpha s)\theta}(\cos\theta)^{c+\lambda s-1}(\sin\theta)^{c+\lambda s-1} \]

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\[
\frac{K}{2\pi i} \int e^{\frac{a}{2}z} \left[ \sum_{j=1}^{m} \prod_{j=1}^{n} \left[ 1 - \alpha_{j} + i\beta_{j} \right] \left( \frac{a + b + \lambda}{2} \right) \right] ds
\]

\[
\int_{0}^{\pi} e^{(\cos\theta + i\theta)r} (\cos\theta) e^{-i(\sin\theta)^c} \left\{ (\theta + a) \frac{a + b + 2}{2} e^{\theta\cos\theta} \right\} d\theta
\]

\[
\sum_{j=1}^{m} \prod_{j=1}^{n} \left[ 1 - \alpha_{j} + i\beta_{j} \right] \left( \frac{a + b + \lambda}{2} \right) \]
And their result is

\[
2^{\alpha-b-2}e^{\frac{e^{\alpha\lambda}}{2}}\left(\sum_{i=0}^{\infty}\frac{a_i}{i!}\frac{b_i}{b_i!}\frac{c_i}{c_i!}\frac{1}{2^{i+2}}\right)\left(\sum_{i=0}^{\infty}\frac{a_i}{i!}\frac{b_i}{b_i!}\frac{c_i}{c_i!}\frac{1}{2^{i+2}}\right)
\]

\[
\left(1-c_\lambda^2\right)\left(\frac{a+b+c+1}{2}\lambda\right)\left(\frac{a_\lambda^2}{a_\lambda^2}\frac{1}{2}\lambda\right)
\]

\[
\left(1-c_\lambda^2\right)\left(\frac{a+b+c+1}{2}\lambda\right)\left(\frac{a_\lambda^2}{a_\lambda^2}\frac{1}{2}\lambda\right)
\]

Provided by: The I-function occurring on both sides as analytic and satisfied the conditions similar as given with the definition of I-function by special cases we get many single integral with hyper geometric function and I-function and their proof according section 3.

5. CONCLUSION

In view of generality of the multivariable H-function, the result obtained in this paper are of a single integral with hyper geometric function and I-function prove to be useful in several interesting situation appearing in the literature on various general transcendental function and it's applications. The generalizations of hyper geometric function and I-function has recently led to many remarkable applications in Statistics, Physics, Number Theory and Combinatory analysis.

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